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Nonlocality and Stochastic Quantization of Field Theory

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The concept of nonlocality is introduced into physics by means of a stochastic context using Langevin and Schwinger-Dyson techniques. This allows us to reformulate the finite theory of quantum fields, free from ultraviolet divergences, based on the stochastic quantization method with nonlocal regulators. As a nonlocal regulator we choose any entire analytic function in the momentum space, which guarantees that our regularization method for any theory of interest does not violate basic physical principles such as unitarity, causality, and gauge invariance of the theory. Here we present the regularization scheme for scalar, gauge, and scalar electrodynamic theories. Our mathematical prescription is similar to the continuum regularization method of quantum field theory with meromorphic regulators investigated by Bern and his team.

1. INTRODUCTION

In recent years, interest has significantly increased in the study of stochastic processes and nonlocal (or extended) objects—fields; this is due to the fact that it has been possible, first, to establish an intimate connection between the theory of stochastic processes and quantum physics (Nelson, 1967; Guerra, 1981; Migdal, 1986; Namsrai, 1986) [see Damgaard and Hüffel (1987) for earlier references], and second, to construct a unified theory of all types of elementary particle interactions, including gravitational force (Furlan *et al.*, 1987; Green *et al.*, 1987; Chaichian and Nelipa, 1984; Lai, 1983; Wali, 1987). The former is known under the general name of the stochastic quantization of systems. There are different approaches to the description of stochastic processes, which formally coincide with quantum phenomena. Among these, the attraction of the stochastic quantization

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method proposed by Parisi and Wu (1981) is that it has succeeded in reducing quantum field theory to a Gaussian stochastic process called the Langevin equation, which usually runs in an auxiliary "fifth time."

Other directions are being developed in the investigation of nonlocal extended objects. Some of them originally arose from intrinsic problems of local quantum field theory, such as the ultraviolet divergences, problems of electron self-energy, etc. To solve these problems, it is usually assumed that the idealized concept of locality may be violated at small distances and some static characteristics of elementary particles must be described by nonlocal values with distributions over space; for example, the charge and mass of the particle may be represented in the form

$$e = \int \rho_e(\mathbf{r})\, d\mathbf{r}; \qquad m = \int \rho_m(\mathbf{r})\, d\mathbf{r}$$

On the other hand, mathematically this means that the Dirac δ-function distribution should be replaced by a nonlocal distribution of the type [for details, see Efimov (1985)]

$$\delta^4(x) \Rightarrow K(x) = \sum_{n=0}^{\infty} \frac{c_n}{(2n)!} (\Box l^2)^n \delta^{(4)}(x) \tag{1.1}$$

or for the wave function of the particle

$$\phi(x) \Rightarrow \varphi(x) = \int (d^4 y) K(x-y) \phi(y) \tag{1.2}$$

[$\phi(x)$ is a local field], i.e., elementary particles may be understood as spread-out (or nonlocal) objects with some dimension l of length (see Figure 1).

It should be noted that from a purely geometrical point of view, a relativistic invariant description of extended objects is possible only in the one-dimensional case, i.e., relativistic dynamics for strings may be success-

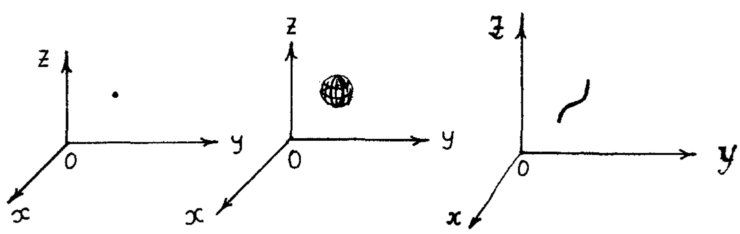

Fig. 1. Illustration of local and nonlocal objects depending on the dimension of space: (a) local object, (b) spread-out (extended) object (ball, bag, etc.) in three-dimensional case, (c) extended object (string) in one-dimensional case.

fully constructed. Nevertheless, from the field point of view, a relativistic invariant construction of the interaction picture between nonlocal objects of type (1.2) is also achieved due to relativistic invariant properties of nonlocal distributions (1.1). In the last case, a basic peculiarity of introducing nonlocality (1.1) is that it leads to a change of the particle propagator, for example, for a scalar particle:

$$\begin{aligned}\Delta(x-y)&=\langle 0|T(\phi(x)\phi(y))|0\rangle\\ &\Rightarrow D(x-y)\\ &=\langle 0|T(\varphi(x)\varphi(y))|0\rangle\\ &=\int\frac{d^4p}{(2\pi)^4 i}\,e^{-ip(x-y)}\,\frac{V(p^2l^2)}{m^2-p^2-i\varepsilon}\end{aligned}\tag{1.3}$$

where $V(p^2l^2)$ is the Fourier transform of the nonlocal distribution $K(x)$.

In this paper, we present a method of introducing nonlocality (1.1)-(1.3) into a stochastic quantization scheme within the framework of the Langevin and Schwinger-Dyson formalisms [for details, see Bern *et al.* (1987*a,b*)]. These two equivalent formulations describe quantum field theory in d dimensions by means of Markovian stochastic processes in $(d+1)$ dimensions via a regularized Parisi-Wu (1981) Langevin equation and by a d-dimensional prescription via regularized Schwinger-Dyson equations, respectively. We assume that the noise term in these equations plays a double role in the theory: it controls the quantum behavior of the theory and at the same time it carries nonlocality in the stochastic equations. Further, we show that the scheme obtained in such a way is equivalent to the nonlocal theory with regularized propagator of the type of (1.3).

An outline of the present paper is as follows. Section 2 introduces the nonlocality into the $(d+1)$-dimensional Langevin formulation for the scalar theory. In Sections 3 and 4 we discuss the equivalent d-dimensional regularized Schwinger-Dyson equations and their more or less conventional weak coupling expansion applied to the calculation of the three-point junction. Section 5 is devoted to the introduction of nonlocality into gauge theory and to the reformulation of gauge-covariant Langevin systems in $(d+1)$ dimensions, for which we derive the regularized Langevin-Feynman rules. These rules are applied in Section 6 to a computation of the one-loop gluon mass in QCD_4. As sketched in Bern *et al.* (1987*a-c*), the mass is zero, providing an explicit check of gauge invariance of this order for entire analytic regulators. Section 7 deals with the simplest gauge theory scalar electrodynamics. This last section is preparative in character in order to generalize our prescription to the non-Abelian theory and the serious student may be advised to begin with this case.

2. NONLOCAL GAUSSIAN NOISE AND REGULARIZED LANGEVIN SYSTEMS FOR THE SCALAR THEORY

2.1. Nonlocal Noise

We consider the Markovian Parisi–Wu Langevin system for a d-dimensional theory of a scalar local field $\phi(x)$ with Euclidean action S:

$$\dot{\phi}(x, t) = -\frac{\delta S}{\delta \phi}(x, t) + \eta(x, t) \tag{2.1}$$

where t is an additional fictitious "fifth-time" variable, x represents d-dimensional Euclidean coordinates, and $\eta(x, t)$ is the usual local Gaussian noise satisfying the condition

$$\langle \eta(x, t)\eta(y, \tau)\rangle_\eta = 2\delta(t-\tau)\delta^d(x-y) \tag{2.2}$$

Now the question arises of how to introduce nonlocality into this stochastic equation in order to obtain an equivalent stochastic formulation for the nonlocal field $\varphi(x)$ in (1.2) with propagator (1.3) in the Euclidean metric. We assume that the noise term in (2.1) carries nonlocality only and by analogy with (1.2), in this case, it takes the form

$$\eta(x, t) \Rightarrow \Lambda(x, t) = \int (dy)\, K(x-y)\eta(y, t) \tag{2.3}$$

where $(dy) = d^d y$, and $K(x)$ is nonlocal distribution investigated in detail by Efimov (1977, 1985). The nonlocal distribution $K(x-y) = K_{xy}(\Box)$ that multiplies the noise is a function of the Laplacian

$$\begin{gathered} \Box_{xy} = \int (dz)(\partial_\mu)_{xz}(\partial_\mu)_{zy} \\ (\partial_\mu)_{xy} \equiv \partial_\mu^x\, \delta^d(x-y) \end{gathered} \tag{2.4}$$

which guarantees that $K_{xy}(\Box) = K_{yx}(\Box)$. We will choose here a wide class of distributions

$$K_{xy}(\Box) = \sum_{n=0}^{\infty} \frac{c_n}{(2n)!}(\Box l^2)^n \delta^d(x-y) \tag{2.5}$$

for which the ordinary Parisi–Wu equation is regained in the limit $l \to 0$, i.e., $K_{xy}(\Box) \to_{l\to 0} \delta^d(x-y)$.

2.2. Nonlocal Distributions

We see that the function (2.5) is the generalized form of the well-known local Dirac δ-function. As usual, its space-time properties are investigated in Minkowski space-time with metric $g_{\mu\nu}=(g_{00}=-g_{11}=-g_{22}=-g_{33}=1$, $g_{\mu\nu}=0$, $\mu\neq\nu)$ and depend essentially on the sequence of coefficients c_n (generally speaking, they are complex numbers). We say that the generalized function (2.5) is given in some test function space if for any $f\in\mathcal{U}$ the functional

$$(K,f)=\int d^dx\,K(x)f(x)=\sum_{n=0}^{\infty}\frac{c_n l^{2n}}{(2n)!}\Box^n f(x)|_{x\to 0}<\infty \tag{2.6}$$

is well defined, i.e., the obtained series converges absolutely. Passing to the momentum space in (2.6), we obtain

$$(K,f)=\int d^dp\,\tilde{K}(p^2)\tilde{f}(p)<\infty \tag{2.7}$$

where

$$\tilde{K}(p^2)=\sum_{n=0}^{\infty}\frac{c_n l^{2n}}{(2n)!}(p^2)^n \tag{2.8}$$

and $\tilde{f}(p)$ is the Fourier transform of $f(x)$. In other words, the generalized function (2.5) is given on $\mathcal{U}$ if the series (2.8) defines the function $\tilde{K}(p^2)$ for all p^2 and the integral (2.7) converges for any $f(x)\in\mathcal{U}$. Both conditions (2.6) and (2.7) are equivalent.

As shown by Efimov (1977), basic physical principles, such as unitarity and causality, dictate that as the Fourier transform of (2.5) an entire analytic function should be chosen. Further, we are interested only in the class of distributions $K(x)$ for which $\tilde{K}(z)$ in (2.8) are entire functions of the variable z with a finite order of growth, $\infty>\rho\geq\frac{1}{2}$, and which decrease rapidly enough when $z=p^2\to-\infty$ (in the Euclidean direction).

In the Euclidean domain of the variable p^2 for the Fourier transform (2.8), the Mellin representation

$$\tilde{K}(-p_E^2l^2)=\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty}d\xi\frac{w(\xi)}{\sin\pi\xi}\,l^{2\xi}(m^2+p_E^2)^{\xi} \tag{2.9a}$$

or

$$V(-p_E^2l^2)=[\tilde{K}(-p_E^2l^2)]^2=\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty}d\xi\frac{v(\xi)}{\sin\pi\xi}\,l^{2\xi}(m^2+p_E^2)^{\xi} \tag{2.9b}$$

$(0<\beta<2)$ is valid. The form of the functions $w(\xi)$ and $v(\xi)$ depends on the form of the function $K(-p_E^2l^2)$. For example, if

$$V_1=\frac{m^4l^4b^{-4}}{[\sin(ml)/ml-\cos(ml)]^2}\left(\frac{\sin b}{b}-\cos b\right)^2,\qquad k_2=\frac{\sin^2 b}{b^2}$$
$$V_2=[(\sin b)/b]^4;\qquad V_3=e^{-b^2};\qquad V_4=2^s\Gamma(1+s)\frac{J_s(b)}{b^s} \tag{2.9c}$$

where $J_s(u)$ is the Bessel function for some given value $s>0$, and $b=[(m^2+p_E^2)l^2]^{1/2}$, then

$$\begin{aligned} v_1(\xi) &= 9\cdot 2^{4+2\xi}(2\xi^2+7\cdot\xi+5)/\Gamma(7+2\xi) \\ v_2(\xi) &= 2^{3+2\xi}(2^{2\xi+2}-1)/\Gamma(5+2\xi) \\ w_2(\xi) &= 2^{1+2\xi}/\Gamma(3+2\xi) \\ v_3(\xi) &= 1/\Gamma(1+\xi) \\ v_4(\xi) &= \Gamma(1+s)/[2^{2\xi}\Gamma(1+\xi)\Gamma(1+s+\xi)] \end{aligned} \tag{2.9d}$$

The physical meaning of the form factor $V(-p_E^2l^2)$ consists in changing the form of the potentials between interacting fields (for example, the Coulomb and Yukawa laws) at small distances and in making the theory finite in each order of the perturbation series of the theory of the coupling constant (Efimov, 1977; Namsrai, 1986). The question of the possible unique choice of the form factors was discussed by Efimov (1977) (see also Papp, 1975). Efimov (1977) has shown that the objects constructed by the distributions $K(x)$ of (2.5) are spread out (nonlocalized) over space. Thus, the relativistic invariant distributions $K(x)$ give a correct description of extended objects. In this case, roughly speaking, the parameter l may be identified with the size of an extended object (a particle).

Our next goal is to introduce such a type of nonlocality into stochastic equations. We now turn to this problem.

2.3. Regularized Langevin Systems for the Scalar Theory

With the assumption (2.3), equation (2.1) now acquires the form

$$\dot{\phi}(x,t)=-\frac{\delta S}{\delta\phi}(x,t)+\int (dy)\,K(x-y)\eta(y,t) \tag{2.10}$$

An expression of type (2.10) makes it possible to realize our program mentioned in a previous work (Namsrai, 1986). We notice that our stochastic prescription using entire analytic regulators including exponential ones may be technically superior and useful for nonperturbative analysis, which appeared already in a paper due to Doering (1985) using the scalar prototype regulator described by Bern *et al.* (1987*a*). As in the usual local stochastic formulation, our prescription for the nonlocal Euclidean Green functions of the theory

$$\langle F[\phi(\cdot)]\rangle=\lim_{t\to\infty}\langle F[\phi(\cdot,t)]\rangle_\eta \tag{2.11}$$

completes the computational scheme.

According to Bern *et al.* (1987*a*), the method expounded in this section is easily generalized for a local symmetry, which will be discussed in Section 5. In this case, the only change in the scheme is the replacement of the Laplacian by the covariant Laplacian in equations (2.1)-(2.5) and (2.10).

We will further follow Bern *et al.* (1987*a,b*) everywhere and obtain explicit weak coupling expressions for equation (2.10). First consider the simpler case

$$S=\int (dx)[\tfrac{1}{2}(\partial_\mu\phi)(\partial_\mu\phi)+\tfrac{1}{2}m^2\phi^2+\lambda(\phi)] \tag{2.12}$$

To solve equation (2.10) with (2.12) and calculate correlation functions in the free case, it is convenient to introduce the free Green's function $G(x, t)$, which satisfies

$$\frac{\partial}{\partial t}G(x,t)-(\Box-m^2)G(x,t)=\delta^d(x)\delta(t)$$

with the initial condition

$$G(x,t)=0, \qquad t<0$$

This equation is easily solved to give the explicit expression for G:

$$G(x,t)=\Theta(t)\int (dp)e^{-ipx-t(p^2+m^2)} \tag{2.13}$$

where $(dp)=d^dp/(2\pi)^d$ and $p\equiv p_E$. Thus, for (2.12), the integral formulation of the system (2.10) is

$$\phi(x,t)=\int (dy)\int_{-\infty}^{t'} dt'\, G(x-y,t-t')$$
$$\times\left[\int (dx_1)\, K_{yx_1}(\Box)\eta(x_1,t')-\lambda'(\phi(y,t'))\right] \tag{2.14}$$

Here λ' is the first derivative of the potential and we have employed the technical device of choosing $t_0=-\infty$, so that the system has equilibrated at any finite fifth time. The integral equation may be iterated to any desired order (Parisi and Wu, 1981) as

$$\phi(x,t)=\int_1 G_{x1}(K\eta)_1-\int_1 G_{x1}\lambda'\left(\int_2 G_{12}(K\eta)_2-\int_2 G_{12}\lambda'\left(\int_3 G_{23}(K\eta)_3\ldots\right)\right) \tag{2.15}$$

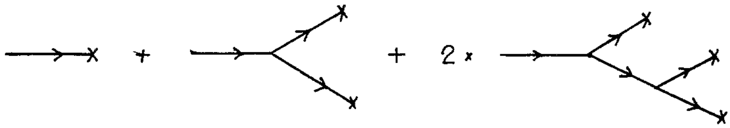

Fig. 2. Langevin tree diagrams through $O(g^2)$ in the nonlocal stochastic scheme.

where we used the compact notation

$$G_{x1} \equiv G(x - x_1, t - t_1)$$

$$(K\eta)_1 \equiv \int (dy)\, K_{x_1 y}(\Box)\eta(y, t_1) \tag{2.16}$$

$$\int_1 \equiv \int (dx_1)\, dt_1$$

According to Bern *et al.* (1987*a*), for concrete calculation purpose it is convenient to represent this iteration by Langevin "tree diagrams," as shown in Figure 2 for the explicit choice $\lambda = g\phi^3/3!$. In these diagrams, each line corresponds to a Langevin Green function (2.13), and its arrow represents its retarded property, while a cross at the end of a line represents a nonlocal form factor (or regulator) times a noise factor.

In the nonlocal stochastic scheme, the tree diagrams may be succinctly summarized in a simple set of Langevin tree rules, as shown for this case in figure 3.

Using equations (2.2), (2.13), and (2.15), we easily obtain correlation functions for the free case $g = 0$,

$$\begin{aligned} &D(x - y, t_1 - t_2) \\ &\quad = \langle \phi(x, t_1)\phi(y, t_2)\rangle_\eta \\ &\quad = 2 \int (dx_1)(dy_1) \int_{-\infty}^{\min(t_1, t_2)} d\tau\, G(x - x_1, t_1 - \tau) G(y - y_1, t_2 - \tau) \\ &\quad \times \int (dz_1)\, K_{x_1 z_1}(\Box) K_{y_1 z_1}(\Box) \end{aligned}$$

Taking into account the obvious equalities

$$\int (dz_1)\, K_{x_1 z_1}(\Box) K_{z_1 y_1}(\Box) = \int (dq)\, e^{-iq(x_1 - y_1)}\, V(-q^2 l^2)$$

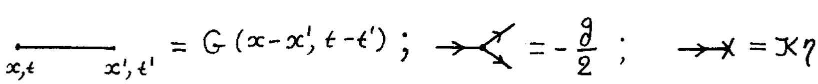

Fig. 3. Langevin tree rules for the nonlocal stochastic quantization theory.

and

$$\int_{-\infty}^{\min(t_1,t_2)} d\tau \exp\{-(t_1-\tau)(p^2+m^2)-(t_2-\tau)(p^2+m^2)\}$$
$$=\frac{\exp\{-|t_1-t_2|(p^2+m^2)\}}{2(m^2+p^2)}$$

we get

$$D_E(x-y)=\lim_{t_1\to t_2} D(x-y,\, t_1-t_2)=\int (dp)\,\frac{V(-p^2l^2)}{m^2+p^2}\, e^{-ip(x-y)} \tag{2.17}$$

which is just the nonlocal Euclidean Green function (1.3) for the scalar theory. Here we have used the definitions

$$K_{xy}(\Box)=\int (dp)\, e^{-ip(x-y)}\, K(-p^2l^2); \qquad V(-p^2l^2)=[K(-p^2l^2)]^2$$

This result also may be obtained by using the diagrammatic representation for the Langevin system. Thus, as a specific example, the zeroth-order momentum-space nonlocal two-point function shown in Figure 4 contains two local Langevin Green functions in the combination

$$D_{12}^l(p)=2V(-p^2l^2)\int_{-\infty}^{t_1} dt_3 \int_{-\infty}^{t_2} dt_4\, G_{13}(p)G_{24}(p)\delta(t_3-t_4)$$
$$=\frac{V(-p^2l^2)}{\Delta_p}\, e^{-|t_1-t_2|\Delta p}=D(p)\exp(-|t_1-t_2|\Delta_p) \tag{2.18a}$$

where we have introduced

$$D(p)=V(-p^2l^2)/\Delta_p, \qquad \Delta_p\equiv p^2+m^2$$
$$G_{ij}(p)=\Theta(t_i-t_j)\, e^{-(t_i-t_j)\Delta_p} \tag{2.18b}$$

The result for the nonlocal free propagator is therefore

$$\langle\phi(x_1)\phi(x_2)\rangle^{(0)}=\int (dp)\, e^{-ip(x_1-x_2)}\, D_{00}^l(p)=\int (dp)\, e^{-ip(x_1-x_2)}\,\frac{V(-p^2l^2)}{m^2+p^2}$$

or

$$\langle\phi_{p_1}\phi_{p_2}\rangle^{(0)}=\frac{V(-p_1^2l^2)}{\Delta_{p_1}}\,\bar\delta^d(p_1+p_2)=D(p_1)\bar\delta^d(p_1+p_2) \tag{2.18c}$$

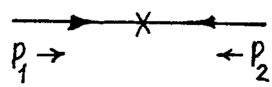

Fig. 4. Langevin line with a contraction in the nonlocal case.

where

$$\bar{\delta}^d(p_1+p_2)\equiv(2\pi)^d\delta^d(p_1+p_2); \qquad \phi(x)=\int (dp)\,\phi_p\,e^{-ipx} \quad (2.18\text{d})$$

In general, each line with a cross (contraction) in a Langevin diagram is represented by a factor $D_{12}^l(p)$, which includes a factor $V(-p^2l^2)$. In this connection, it should be noted that product of generalized functions $K_{zy}(\Box)$ may be understood as a contraction operation only. For example,

$$K_{xy}^2(\Box)=\int (dz)\,K_{xz}(\Box)K_{zy}(\Box) \quad (2.19)$$

or

$$\Box_{xy}^2=\int (dz)\,\Box_{xz}\Box_{zy}$$

etc.

For further calculation experience, we consider ϕ^3 theory and calculate the nonlocal first-order three-point function (Figure 5).

Let

$$\lambda(\phi)=g\phi^3/3!$$

In this concrete case, iteration solution (2.15) takes the form in the momentum representation

$$\phi(x,t)=\int (dp)\,e^{-ipx}\,\tilde{\phi}_p(t)$$

where

$$\begin{aligned}\tilde{\phi}_p(t)=\int (dx')\int_{-\infty}^{t} dt'\,e^{ipx'}\,G_{tt'}(p)\bigg\{\int (dy)\,K_{x'y}(\Box)\eta(y,t')\\ -\frac{g}{2}\int (dx_1)\int_{-\infty}^{t'} dt_1\,G(x'-x_1,t'-t_1)\int (dy_1)\,K_{x_1y_1}(\Box)\eta(y_1,t_1)\\ \times\int (dx_2)\int_{-\infty}^{t'} dt_2\,G(x'-x_2,t'-t_2)\int (dy_2)\,K_{x_2y_2}(\Box)\eta(y_2,t_2)\bigg\}\end{aligned} \quad (2.20)$$

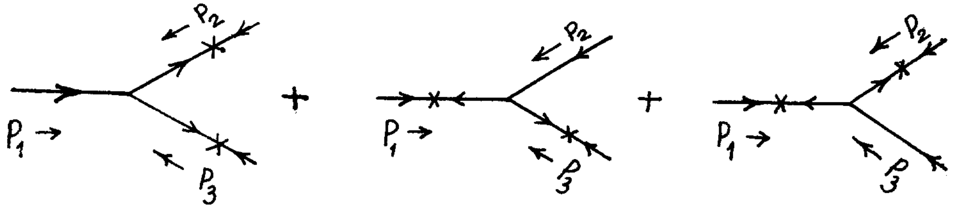

Fig. 5. Langevin three-point diagrams in the nonlocal stochastic case.

To calculate $\langle\phi_{p_1}\phi_{p_2}\phi_{p_3}\rangle^{(1)}_{\text{conn}}$ for connected diagrams, we use the approximation

$$\left(a_x-\frac{g}{2}b_x\right)\left(a_y-\frac{g}{2}b_y\right)\left(a_z-\frac{g}{2}b_z\right)=a_xa_ya_z-\frac{g}{2}(b_xa_ya_z+b_ya_xa_z+b_za_xa_y)$$

and the Gaussian noise property

$$\begin{aligned}\langle\eta(x_1,t_1)\eta(x_2,t_2)\eta(x_3,t_3)\eta(x_4,t_4)\rangle&\\=4[\delta^d(x_1-x_2)\delta(t_1-t_2)\delta^d(x_3-x_4)\delta(t_3-t_4)&\\+\delta^d(x_1-x_3)\delta(t_1-t_3)\delta^d(x_2-x_4)\delta(t_2-t_4)&\\+\delta^d(x_1-x_4)\delta(t_1-t_4)\delta^d(x_2-x_3)\delta(t_2-t_3)]&\end{aligned}\tag{2.21}$$

After integration over t_i and x_i variables, we have

$$\begin{aligned}\langle\phi_{p_1}\phi_{p_2}\phi_{p_3}\rangle^{(1)}_{\text{conn}}&\\=-g\int dt_1\,[G_{01}(p_1)D^l_{01}(p_2)D^l_{01}(p_3)+D^l_{01}(p_1)G_{01}(p_2)&\\\times D^l_{01}(p_3)+D^l_{01}(p_1)D^l_{01}(p_2)G_{01}(p_3)]\bar{\delta}^d(p_1+p_2+p_3)&\end{aligned}\tag{2.22}$$

Taking into account the explicit forms (2.18a) and (2.18b) for $D^l_{ij}(p)$ and $G_{ij}(p)$ functions and carrying out some algebraic operations, we get

$$\langle\phi_{p_1}\phi_{p_2}\phi_{p_3}\rangle=-g\frac{\sum_{i=1}^3(V^{-1}\Delta)_{p_i}}{\sum_{i=1}^3\Delta_{p_i}}\prod_{i=1}^3(V\Delta^{-1})_{p_i}\bar{\delta}^d(p_1+p_2+p_3)\tag{2.23}$$

We note that in the presence of the form factor, the loop (Figure 6)

$$\begin{aligned}\langle\phi_p\rangle^{(1)}&=-\tfrac{1}{2}g\int dt_1\,G_{01}(p)\int(dk)\,D^l_{11}(k)\,\bar{\delta}^d(p)\\&=-\tfrac{1}{2}g\frac{\bar{\delta}^d(p)}{\Delta_p}\int(dk)\,D(k);\qquad D(k)=V(-k^2l^2)/\Delta_k\end{aligned}\tag{2.24}$$

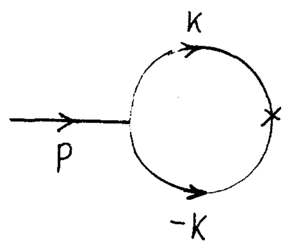

Fig. 6. Langevin tadpole diagram in the nonlocal stochastic scheme.

is not the proper vertex of (2.23) times a nonlocal propagator. This indicates some peculiar of the effective d-dimensional action of the theory, which will be discussed in Sections 3 and 4.

3. NONLOCAL SCHWINGER–DYSON EQUATIONS

3.1. Derivation of the SD Equations

The regularized Schwinger–Dyson (SD) equations with meromorphic regulators were used in the stochastic quantization scheme due to Bern *et al.* (1987*a*,*b*). We generalize here their results for a wide class of nonlocal distributions, the Fourier transforms of which are entire analytic functions of the type (2.9a). It is shown that a simple d-dimensional SD formulation depends crucially on the Markovian property of the scheme at the stochastic level. It turns out that this property does not change in our case.

We begin with the Langevin system (2.10) and (2.12). Let $F[\phi]$ be any equal fifth-time functional of the field ϕ; then its η average evolves in fifth time according to

$$\frac{\langle dF[\phi]\rangle_\eta}{dt}=\left\langle\int(dx)\,\frac{\partial\phi(x,t)}{\partial t}\,\frac{\delta F[\phi]}{\delta\phi}\right\rangle_\eta \tag{3.1}$$

To transform this equation, we use the local white noise identity

$$\left[\eta(y,t)+2\frac{\delta}{\delta\eta(y,t)}\right]\exp\left[-\tfrac{1}{4}\int d\tau\,(dx)\,\eta^2(x,\tau)\right]=0 \tag{3.2}$$

which expresses the Markovian property of our scheme and is easily verified by differentiating $\exp[-\frac{1}{4}\int d\tau\,(dx)\,\eta^2(x,\tau)]$ with respect to $\eta(y,t)$. Thus, multiplying (3.2) by any functional $F(\phi)$ and integrating it over η, we get

$$\int_{-\infty}^{\infty}d\eta\left[\eta(y,t)+2\frac{\delta}{\delta\eta(y,t)}\right]\exp\left[-\tfrac{1}{4}\int d\tau\,(dx)\,\eta^2(x,\tau)\right]F(\phi)=0$$

Integration by parts in η gives

$$\int_{-\infty}^{\infty}d\eta\,\exp\left[-\tfrac{1}{4}\int d\tau\,(dx)\,\eta^2(x,\tau)\right]\left[\eta(y,t)-2\frac{\delta}{\delta\eta(y,t)}\right]F(\phi)=0$$

from which follows the formal definition

$$\eta(y,t)=2\frac{\delta}{\delta\eta(y,t)}=2\int(dz)\,\frac{\delta\phi(z,t)}{\delta\eta(y,t)}\,\frac{\delta}{\delta\phi(z,t)} \tag{3.3}$$

for any functional $F(\phi)$. Now it is necessary to define $\delta\phi(x,t)/\delta\eta(y,t)$.

For this, using the Langevin equation and its free solution, we obtain

$$\begin{aligned}\frac{\delta\phi(x,t)}{\delta\eta(y,t)}&=\frac{\delta}{\delta\eta(y,t)}\int(dx')\int_{-\infty}^{t}dt'\,G(x-x',t-t')\int(dz)\,K_{x'z}(\square)\eta(z,t')\\&=\int(dx')\int_{-\infty}^{t}dt'\int(dp)\,e^{-i(x-x')p}\,G_{tt'}(p)K_{x'y}(\square)\delta(t-t')\\&=\Theta(0)K_{xy}(\square)=\tfrac{1}{2}K_{xy}(\square)\end{aligned}\tag{3.4}$$

Further, according to equalities (3.3) and (3.4), we get a chain rule into $\delta/\delta\phi$:

$$\begin{aligned}&\int(dy)\,K_{xy}(\square)\eta(y,t)\\&\quad=2\int(dy)\,K_{xy}(\square)\int(dz)\frac{\delta\phi(z,t)}{\delta\eta(y,t)}\frac{\delta}{\delta\phi(z,t)}\\&\quad=\int(dy)\,K_{xy}(\square)\int(dz)\,K_{yz}(\square)\frac{\delta}{\delta\phi(z,t)}\\&\quad=\int(dz)\,K^2_{xz}(\square)\frac{\delta}{\delta\phi(z,t)}\end{aligned}\tag{3.5}$$

where, by definition (2.19),

$$K^2_{xy}(\square)=\int(dz)\,K_{xz}(\square)K_{zy}(\square)$$

or

$$\int(dz)\,K_{xz}(\square)K_{zy}(\square)=\int(dp)\,V(-p^2l^2)\,e^{-ip(x-y)}\equiv K^2_{xy}(\square)$$

Finally, taking into account (2.10) and (3.1)-(3.5), we arrive at the definition for the regularized SD equations

$$\left\langle\frac{d}{dt}F(\phi)\right\rangle_\eta=\left\langle\int(dx)\left[-\frac{\delta S}{\delta\phi(x)}+\int(dy)\,K^2_{xy}(\square)\frac{\delta}{\delta\phi(y)}\right]\frac{\delta F(\phi)}{\delta\phi(x)}\right\rangle_\eta\tag{3.6}$$

or, at equilibrium,

$$\left\langle\int(dx)\left[-\frac{\delta S}{\delta\phi(x)}+\int(dy)\,K^2_{xy}(\square)\frac{\delta}{\delta\phi(y)}\right]\frac{\delta F(\phi)}{\delta\phi(x)}\right\rangle=0\tag{3.7}$$

Further, following Bern *et al.* (1987*a*) and choosing

$$F(\phi)=\exp\left\{\int(dx)\,J(x)\phi(x)\right\}$$

we can easily obtain the Schwinger form of these equations,

$$\int(dx)\,J(x)\left[-\frac{\delta S}{\delta\phi(x)}\bigg|_{\phi\to\delta/\delta J}+\int(dy)\,K^2_{xy}(\square)J(y)\right]Z(y)=0\tag{3.8}$$

where $Z(J)=\langle\exp\{\int(dx)\,J(x)\phi(x)\}\rangle$ is the vacuum-to-vacuum generating functional.

As shown below, the Schwinger–Dyson equations, plus some boundary condition which requires the permutation symmetry of Euclidean Bose time-ordered product, e.g.,

$$\langle\phi_1\phi_2\rangle=\langle\phi_2\phi_1\rangle \qquad (3.9)$$
$$\cdots$$

are equivalent (at least in the weak coupling limit) to the Langevin formulation at equilibrium.

It is convenient to study the SD equations (3.7) in momentum space. Making use of the definitions (2.18b) and (2.18d) and the simple relations

$$\frac{\delta}{\delta\phi(x)}=\int(dp)\,e^{+ipx}\frac{\delta}{\delta\phi_p};\qquad \frac{\delta\phi_p}{\delta\phi_k}=\bar{\delta}^d(p+k);$$

$$\frac{\delta F(\phi)}{\delta\phi(x)}=\int(dp_2)\,e^{ip_2x}\frac{\delta F(\phi_p)}{\delta\phi_{p_2}}$$

we have the following identities:

$$\int(dx)(\partial^2-m^2)\phi(x)\frac{\delta F(\phi)}{\delta\phi(x)}=-\int(dp)(p^2+m^2)\phi_p\frac{\delta F(\phi)}{\delta\phi_p}$$

$$\int(dx)(dy)\,K^2_{xy}(\Box)\frac{\delta^2F(\phi)}{\delta\phi(y)\,\delta\phi(x)}=\int(dp)\,V(-p^2l^2)\frac{\delta^2F(\phi)}{\delta\phi_p\,\delta\phi_{-p}}$$

etc., from which it is easily verified by a functional chain rule that

$$\left\langle\int(dp)\,\Delta_p\phi_p\frac{\delta F}{\delta\phi_p}\right\rangle$$
$$=\left\langle\int(dp)\,V(-p^2l^2)\frac{\delta^2F}{\delta\phi_p\,\delta\phi_{-p}}\right.$$
$$\left.-\frac{g}{(N-1)!}\int\prod_{i=1}^{N-1}(dk_i)\,(dp)\,\bar{\delta}^d\left(p-\sum_{i=1}^{N-1}k_i\right)\phi_{k_1}\cdots\phi_{k_{N-1}}\frac{\delta F}{\delta\phi_p}\right\rangle \qquad (3.10)$$

where we have chosen the interaction

$$\lambda(\phi)=g\phi^N/N!$$

As a first trivial example, with the boundary condition (3.9) we compute the regularized free two-point function. Setting $g=0$ and choosing $F=\phi_{p_1}\phi_{p_2}$, we find that equation (3.10) becomes

$$\langle\phi_{p_1}\phi_{p_2}\rangle^{(0)}=\bar{\delta}^d(p_1+p_2)D(p_1);\qquad D(p)=V(-p^2l^2)\Delta_p^{-1} \qquad (3.11)$$

This result is the correct nonlocal free propagator, in agreement with the Langevin result (2.18c).

3.2. Iterative Procedure for the Nonlocal SD Equations

To compute some n-point functions for any desired order of coupling constant g within the SD equations, the iterative method of equation (3.10) should be given. This procedure was used by Bern *et al.* (1987*a–c*). In our case with nonlocal form factors, their result is automatically carried over. For example, it is not difficult to check in analogy with the formula (3.11) that $\langle\phi_{p_1}\phi_{p_2}\cdots\phi_{p_N}\rangle^{(0)}$ yields the usual Wick expansion, as products of nonlocal free propagators (3.11). Moreover, in the first order of g it corresponds to the regularized vertex

$$\begin{aligned}\Gamma(p_1,\ldots,p_N)&=\langle\phi_{p_1}\cdots\phi_{p_N}\rangle^{(1)}\\&=\delta^d\left(\sum_{i=1}^N p_i\right)\prod_{i=1}^N D(p_i)\frac{(-g)\sum_{j=1}^N[D(p_j)]^{-1}}{\sum_{j=1}^N\Delta_{p_j}}\end{aligned}\tag{3.12}$$

For $N=3$, the result agrees with equation (2.23).

The iterative chain rule may be obtained using equation (3.10). To illustrate this, we consider ϕ^3 theory ($N=3$). First, setting $F(\phi)=\phi_p$ in equation (3.10), we get

$$\langle\phi_p\rangle=-\frac{g}{2}\frac{1}{\Delta_p}\int(dk_1)(dk_2)\,\bar\delta^d(p-k_1-k_2)\langle\phi_{k_1}\phi_{k_2}\rangle\tag{3.13}$$

In turn, $\langle\phi_{k_1}\phi_{k_2}\rangle$ is given by the formula

$$\begin{aligned}\langle\phi_{k_1}\phi_{k_2}\rangle=\bar\delta^d(k_1+k_2)D(k_1)&-\frac{g}{2(\Delta_{k_1}+\Delta_{k_2})}\int(dq_1)(dq_2)\\&\times[\bar\delta^d(k_1-q_1-q_2)\langle\phi_{k_2}\phi_{q_1}\phi_{q_2}\rangle\\&+\bar\delta^d(k_2-q_1-q_2)\langle\phi_{k_1}\phi_{q_1}\phi_{q_2}\rangle]\end{aligned}\tag{3.14}$$

Further, assuming $F(\phi)=\phi_{p_1}\phi_{p_2}\phi_{p_3}$ in equation (3.10), we obtain

$$\begin{aligned}\langle\phi_{p_1}\phi_{p_2}\phi_{p_3}\rangle=&\left[\frac{2\bar\delta^d(p_1+p_2)V(-p_1^2l^2)\langle\phi_{p_3}\rangle}{\Delta_{p_1}+\Delta_{p_2}+\Delta_{p_3}}+\text{cyclic perm in }\{p\}\right]\\&-\frac{g}{2}\frac{1}{\Delta_{p_1}+\Delta_{p_2}+\Delta_{p_3}}\int(dk_1)(dk_2)\\&\times[\bar\delta^d(p_1-k_1-k_2)\langle\phi_{p_2}\phi_{p_3}\phi_{k_1}\phi_{k_2}\rangle\\&+\text{cyclic perm in }\{p\}]+\cdots\end{aligned}\tag{3.15}$$

where the definition $\langle\phi_{p_1}\phi_{p_2}\rangle\equiv\bar\delta^d(p_1+p_2)\Delta_{p_1}V(-p_1^2l^2)$ has been used.

Using the zeroth-order result (3.11) for D_p, we can immediately obtain the first-order tadpole graph (Figure 6) from (3.13) and (3.14),

$$\langle\phi_p\rangle^{(1)} = -\frac{g}{2}\frac{\bar{\delta}^d(p)}{m^2}\int (dk)\, D(k) \tag{3.16}$$

in agreement with the Langevin result (2.24). After taking the next approximation in equation (3.13), we find that expression (3.16) acquires the form

$$\langle\phi_p\rangle^{(2)} = -\frac{g}{2}\left\{\frac{\bar{\delta}^d(p)}{\Delta_p}\int (dk_1)\, D(k_1) - \frac{g}{\Delta_p}\int (dk_1)\,(dq_1)\,(dq_2)\right.$$
$$\left. \times\, \bar{\delta}^d(p-k_1-q_1-q_2)\frac{\langle\phi_{k_1}\phi_{q_1}\phi_{q_2}\rangle}{\Delta_{k_1}+\Delta_{p-k_1}}\right\}$$

Finally, in order to compute the $O(g^2)$ one-loop contribution to the two-point function (Figure 7), we take into account the second term in (3.14) and put in it the disconnected part of (3.15) with

$$\langle\phi_{q_1}\phi_{q_2}\phi_{k_1}\phi_{k_2}\rangle^{(0)}_{q,k}$$
$$= \langle\phi_{q_1}\phi_{k_1}\rangle\langle\phi_{q_2}\phi_{k_2}\rangle + \langle\phi_{q_1}\phi_{k_2}\rangle\langle\phi_{k_1}\phi_{q_2}\rangle$$
$$= D(q_1)D(q_2)[\bar{\delta}^d(q_1+k_1)\bar{\delta}^d(q_2+k_2) + \bar{\delta}^d(q_1+k_2)\bar{\delta}^d(k_1+q_2)]$$

where the subscript on the right q, k means that we keep only those contributions in which q's contract with k's.

As a result of a little algebra, we obtain

$$\Pi(p) = \frac{g^2}{2}\frac{D(p)}{\Delta_p}\int (dk)\, D(k)D(p-k)\frac{(V^{-1}\Delta)_k + (V^{-1}\Delta)_{p-k} + (V^{-1}\Delta)_p}{\Delta_k+\Delta_{p-k}+\Delta_p} \tag{3.17}$$

which is the usual local loop when $l \to 0$.

Thus, the SD equations (3.7) or (3.13)-(3.15) may be solved iteratively in this manner to any desired order of g. However, the procedure is increasingly tedious. To simplify this prescription, Bert *et al.* (1987*a*–*c*) developed a systematic set of Schwinger–Dyson–Feynman rules instead.

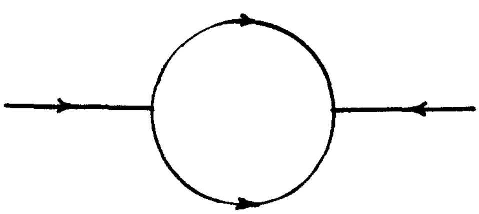

Fig. 7. One-loop two-point function in the nonlocal stochastic scheme.

We mention that the construction of any expressions of the type of (3.17) according to these rules requires more effort than the usual Feynman diagrammatic correspondence.

Finally, for further computational purposes we present here a concrete method for calculating the expression (3.17) the explicit form of which is

$$\Pi(p)=\frac{g^2}{2}\frac{1}{(m^2+p^2)^2}\int (dk)\frac{1}{3m^2+k^2+p^2+(p-k)^2}$$
$$\times\left\{\frac{V(-p^2l^2)V(-(p^2-k)^2l^2)}{m^2+(p-k)^2}+\frac{V(-p^2l^2)V(-k^2l^2)}{m^2+k^2}\right.$$
$$\left.+\frac{(m^2+p^2)V(-k^2l^2)V(-(k-p)^2l^2)}{(m^2+k^2)[m^2+(p-k)^2]}\right\} \tag{3.18}$$

First, consider the second term of (3.18) in the case of $d=6$ dimensions. By using the Mellin representation (2.9b) for $V(z)$ and the general Feynman parametric formula

$$b_1^{-\mu_1}\cdots b_n^{-\mu_n}=\frac{\Gamma(\mu_1+\cdots+\mu_n)}{\Gamma(\mu_1)\cdots\Gamma(\mu_n)}\int_0^1 d\alpha_1\cdots\int_0^1 d\alpha_n\,\delta\left(1-\sum_{i=1}^n\alpha_i\right)\alpha_1^{\mu_1-1}$$
$$\times\alpha_2^{\mu_2-1}\cdots\alpha_n^{\mu_n-1}\left[\sum_{j=1}^n b_j\alpha_j\right]^{-\mu_1-\mu_2-\cdots-\mu_n}$$

we get

$$\Pi^{(2)}(p)=\frac{g^2}{4}\frac{V(-p^2l^2)}{(p^2+m^2)^2}\frac{\pi^3}{(2\pi)^6}\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty}d\xi\frac{v(\xi)}{\sin\pi\xi}\frac{\Gamma(-1-\xi)}{\Gamma(1-\xi)}l^{2\xi}f(\xi) \tag{3.19}$$

where $f(\xi)=\int_0^1 dx\,(1-x)^{-\xi}A^{1+\xi}$, and

$$A=-\tfrac{1}{4}p^2x^2+p^2x+\tfrac{3}{2}m^2x+m^2(1-x)$$

Further, by shifting the contour of integration to the right, we can reduce this integral to a series; taking into account the main asymptotics, we have

$$\Pi^{(2)}(p)=\frac{g^2}{2^9\pi^3}\frac{1}{(m^2+p^2)^2}\left[\frac{\sigma}{l^2}+(\tfrac{5}{6}p^2+\tfrac{5}{2}m^2)v(0)\ln\mu^2l^2\right]$$
$$\sigma=\lim_{x\to-1}v(x)/(1+x) \tag{3.20}$$

where we have assumed that the function $v(x)$ has zero at the point $x=-1$ and $V(-p^2l^2)|_{l^2\to0}=1$ for the external momentum variable p^2. Moreover, in (3.19) we use the Γ-function properties

$$\Gamma(1+x)=x\Gamma(x),\qquad \Gamma(x)\Gamma(1-x)=\pi/\sin\pi x$$

The first and second terms in (3.20) correspond to calculations of residues at points $\xi=-1$ and $\xi=0$, respectively. It is clear that $\Pi^{(1)}(p)=\Pi^{(2)}(p)$.

Similar calculations can be carried out for the third term in (3.18) and the result is reduced to the following formula:

$$\Pi^{(3)}(p) = -\frac{g^2}{2^9\pi^3}\frac{v(0)}{(m^2+p^2)}\ln l^2\mu^2 \tag{3.21}$$

In (3.20) and (3.21), $v(0)=1$, which follows from the normalization condition $V(0)=1$, and μ is an arbitrary parameter with dimension of mass.

4. RENORMALIZATION PRESCRIPTION AND THE THREE-POINT FUNCTION IN NONLOCAL SD FORMALISM

A renormalization program in the regularized SD formalism as was first discussed by Bern *et al.* (1987*a*). For the nonlocal case, their result is immediately repeated. However, a significant difference appears when counterterms in the Lagrangian function are constructed. In the nonlocal stochastic theory counterterms are finite, since we do not assume $l\to 0$ at the end of the calculations. This means that the parameter l of the theory remains everywhere—in particular, in its action. Thus, our scheme is an action regularization, because explicit divergence does not occur in the effective d-dimensional action of the theory and in the Green function expressions.

For completeness, within the SD equations we present here a renormalization procedure due to Bern *et al.* (1987*a*) for the nonlocal case. Thus, the nonlocal SD equations

$$\left\langle \int (dx)\left[\frac{\delta S_0}{\delta\phi(x)} - \int (dy)\, K^2_{xy}(\Box)\frac{\delta}{\delta\phi(y)}\right]\frac{\delta F}{\delta\phi(x)}\right\rangle = 0 \tag{4.1}$$

involve the unrenormalized field $\phi(x)$ and the bare Lagrangian $\mathscr{L}_0$, whose parameters we now denote as m_0 and g_0. The usual renormalized field is $\phi_R \equiv Z_\phi^{-1/2}\phi$, by means of which the renormalized Green functions $F(\phi_R)$ are constructed. Assuming that the SD equations homogeneous in $\delta/\delta\phi$, we have the nonlocal SD equations

$$\left\langle \int (dx)\left[\frac{\delta(S_R+S_{CT})}{\delta\phi_R(x)} - \int (dy)\, K^2_{xy}(\Box)\frac{\delta}{\delta\phi_R(y)}\right]\frac{\delta F(\phi_R)}{\delta\phi_R(x)}\right\rangle = 0 \tag{4.2}$$

where $S_0 = S_R + S_{CT}$ is the usual textbook breakup into the renormalized Lagrangian and the counterterm Lagrangian. The renormalization procedure as usually formulated is based on the construction of the total Lagrangians; for example, in the case of ϕ^3 theory we have explicitly

$$\begin{aligned} \mathscr{L}_R &= \tfrac{1}{2}\phi_R(-\Box+m^2)\phi_R + \frac{g}{3!}\phi_R^3 \\ \mathscr{L}_{CT} &= \tfrac{1}{2}(Z_\phi-1)\phi_R(-\Box+m^2)\phi_R + \tfrac{1}{2}\delta m^2\phi_R^2 + \frac{g}{3!}(Z_j-1)\phi_R^3 \end{aligned} \tag{4.3}$$

where

$$g = Z_\phi^{3/2} g_0 / Z_g, \qquad m^2 = m_0^2 - \delta m^2 / Z_\phi \tag{4.4}$$

Following Bern *et al.* (1987*a*), we compute here three-point vertices in the nonlocal theory using the iterative method presented in the previous section for the SD equations. For this purpose, we continue the iterative procedure carried out in Section 3.2 up to the $O(g^3)$ order for $\langle \phi_{p_1} \phi_{p_2} \phi_{p_3} \rangle_{\text{conn}}$. After simple but tedious calculations, we have

$$\begin{aligned}\langle \phi_{p_1} \phi_{p_2} \phi_{p_3} \rangle = -\frac{g}{2} \rho \int (dk_1)\,(dk_2) \Bigg\{ & \frac{-g\bar{\delta}^d(p_1 - k_1 - k_2)}{2(\Delta_{k_1} + \Delta_{k_2} + \Delta_{p_2} + \Delta_{p_3})} \\ & \times [4\Sigma_1 + 2\Sigma_2 + 4\Sigma_3 + 2\Sigma_4(p_2 \leftrightarrow p_3) + 2\Sigma_5(p_2 \leftrightarrow p_3) \\ & + \Sigma_6(p_2 \leftrightarrow p_3)] + (p_1 \leftrightarrow p_2) + (p_1 \leftrightarrow p_3) \Bigg\} + M_1 + M_2\end{aligned} \tag{4.5}$$

where

$$\rho = \left[\sum_{j=1}^{3} \Delta_{p_j} \right]^{-1}$$

$$\begin{aligned}&\Sigma_i(k_1, k_2; p_2, p_3) \\ &\quad = \left(-\frac{g}{2}\right) \int (dq_1)\,(dq_2)\,(ds_1)\,(ds_2)\,\bar{\delta}^d(k_1 - q_1 - q_2) \\ &\qquad \times [\Delta_{q_1} + \Delta_{q_2} + \Delta_{k_2} + \Delta_{p_2} + \Delta_{p_3}]^{-1} \sigma_i; \qquad i = 1, 2, 3 \\ &\Sigma_j(k_1, k_2; p_2, p_3) \\ &\quad = -\frac{g}{2} \int (dq_1)\,(dq_2)\,(ds_1)\,(ds_2) \sigma_j \bar{\delta}^d(p_2 - q_1 - q_2) \\ &\qquad \times [\Delta_{q_1} + \Delta_{q_2} + \Delta_{k_1} + \Delta_{k_2} + \Delta_{p_3}]^{-1}; \qquad j = 4, 5, 6\end{aligned}$$

Here

$$\begin{aligned}&\sigma_1 = \bar{\delta}^d(q_1 - s_1 - s_2) \langle \phi_{s_1} \phi_{s_2} \phi_{q_2} \phi_{k_2} \phi_{p_2} \phi_{p^3} \rangle^{(0)} \\ &\sigma_2 = \sigma_1(q_1 \leftrightarrow k_2); \qquad \sigma_3 = \sigma_2(k_2 \leftrightarrow p_2) \\ &\sigma_4 = \bar{\delta}^d(q_1 - s_1 - s_2) \langle \phi_{s_1} \phi_{s_2} \phi_{q_2} \phi_{k_1} \phi_{k_2} \phi_{p_3} \rangle^{(0)} \\ &\sigma_5 = \sigma_4(q_1 \leftrightarrow k_2); \qquad \sigma_6 = \sigma_5(k_2 \leftrightarrow p_3)\end{aligned} \tag{4.6}$$

In turn, the terms M_i $(i=1,2)$ are given by the formulas

$$M_1=-g\rho\bar{\delta}^d(p_1+p_2+p_3)$$
$$\times\left\{\left[\frac{V(-p_2^2l^2)}{\Delta_{p_2}+\Delta_{p_3}}\Pi(p_3)+(p_2\leftrightarrow p_3)\right]+(p_1\leftrightarrow p_2)+(p_1+p_3)\right\}$$
$$M_2=\left(-\frac{g}{2}\right)^2\rho\int(dk_1)(dk_2)\gamma_1\left\{\int(dq_1)(dq_2)\right.$$
$$\times\gamma_2\bar{\delta}^d(p_1-k_1-k_2)\bar{\delta}^d(k_1-q_1-q_2)V(-q_1^2l^2)$$
$$\times\left[-\frac{g}{2}\int(ds_1)(ds_2)(H+H(q_2\leftrightarrow p_2)+H(q_2\leftrightarrow p_3)\right.$$
$$+N+N(q_2\leftrightarrow k_2)+N(q_2\leftrightarrow p_3)+L+L(q_2\leftrightarrow k_2)$$
$$\left.\left.+L(q\leftrightarrow p_2))+(q_1\leftrightarrow q_2)\right]+(k_1\leftrightarrow k_2)\right\}$$

Here

$$H=\bar{\delta}^d(q_1+k_2)[\Delta_{p_2}+\Delta_{p_3}+\Delta_{q_2}]^{-1}\bar{\delta}^d(q_2-s_1-s_2)\langle\phi_{s_1}\phi_{s_2}\phi_{p_2}\phi_{p_3}\rangle^{(0)}$$

$$N=2\bar{\delta}^d(q_1+p_2)[\Delta_{k_2}+\Delta_{q_2}+\Delta_{p_3}]^{-1}\bar{\delta}^d(q_2-s_1-s_2)\langle\phi_{s_1}\phi_{s_2}\phi_{k_2}\phi_{p_3}\rangle^{(0)}$$

$$L=N(p_2\leftrightarrow p_3);\qquad \gamma_1=[\Delta_{k_1}+\Delta_{k_2}+\Delta_{p_2}+\Delta_{p_3}]^{-1}$$

$$\gamma_2=[\Delta_{q_1}+\Delta_{q_2}+\Delta_{k_2}+\Delta_{p_2}+\Delta_{p_3}]^{-1}$$

The main asymptotics of (4.5) may be easily calculated by the same method as used in previous sections. We are interested only in divergent parts in the expression (4.5). For example, the term Σ_4 has the form

$$\Sigma_4=8\left(-\frac{g}{2}\right)^3(m^2+p_3^2)^{-1}(m^2+p_2^2)^{-1}\rho\bar{\delta}^d(p_1+p_2+p_3)$$
$$\times\frac{1}{2(2m^2+p_2^2+p_3^2)}\int(dq)\frac{V(-q^2l^2)}{m^2+q^2}\{[\Delta_q+\Delta_{q-p_2}+\Delta_{p_2}+\Delta_{p_3}]^{-1}$$
$$-[\Delta_q+\Delta_{q-p_3}+\Delta_{p_3}+2\Delta_{p_2}]^{-1}\}$$

where we have used the usual Wick expansion for σ_4 in (4.6) in accordance with (3.11). Integration over d^6q is easily carried out by the same prescription presented for obtaining leading terms of the two-loop function $\Pi(p)$

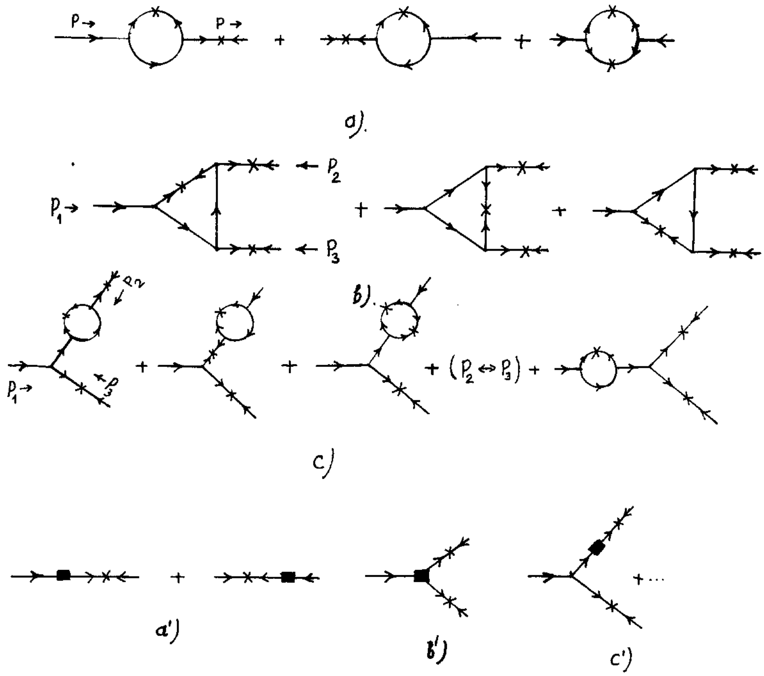

Fig. 8. Nonlocal diagrams. (a) One-loop two-point functions. (b) "Pure" three-point vertices that are infinite as $l \to 0$. (c) Three-point functions with a loop on the external lines. Cyclic permutations of the external lines must also be included. (a'), (b'), (c') respective counterterm diagrams.

Table I

Diagram	Leading terms in sum of one-loop diagrams
Fig. 8a	$\frac{g^2}{2^8\pi^3}\left[\sigma l^{-2}+\left(\frac{p^2}{3}+2m^2\right)\ln \mu^2 l^2\right]\Big/(p^2+m^2)^2$
Fig. 8b	$\frac{g^3}{2^7\pi^3}[(m^2+p_1^2)(m^2+p_2^2)(m^2+p_3^2)]^{-1}\ln \mu^2 l^2$
Fig. 8c	$-\frac{g^2}{2^8\pi^3}[(m^2+p_1^2)(m^2+p_2^2)(m^2+p_3^2)]^{-1} \times\left[\frac{\sigma l^{-2}+(\frac{1}{3}p_1^2+2m^2)\ln \mu^2 l^2}{m^2+p_1^2}+\frac{\sigma l^{-2}+(\frac{1}{3}p_2^2+2m^2)\ln \mu^2 l^2}{m^2+p_2^2}+\frac{\sigma l^{-2}+(\frac{1}{3}p_3^2+2m^2)\ln \mu^2 l^2}{m^2+p_3^2}\right]$

in (3.18). After some elementary calculations, the main asymptotics are reduced to the following formula:

$$\Sigma_4 = -\frac{g^2}{2^9\pi^3}\frac{(2m^2+p_2^2+p_3^2)^{-1}}{(m^2+p_2^2)(m^2+p_3^2)}\rho\left\{\frac{2\sigma}{l^2}+[\tfrac{11}{6}(p_2^2+p_3^2)+7m^2]\ln\mu^2l^2\right\}$$

The remaining terms in (4.5) are calculated in the same manner. According to Bern *et al.* (1987*a*), the results may be classified within the different types of diagrams shown in Figure 8 [for details see Bern *et al.* (1987*a*)].

Final results are given in Tables I and II. Comparing the sum of the loop diagrams in Table I with the sum of the counterterm diagrams in Table II, we determine the renormalization constants

$$\begin{aligned} Z_\phi &= 1+\frac{1}{3}\frac{g^2}{2^8\pi^3}\ln\mu^2l^2 \\ Z_g &= 1+\frac{g^2}{2^7\pi^3}\ln\mu^2l^2 \\ \delta m^2 &= \frac{g^2}{2^8\pi^3}(\sigma l^{-2}+\tfrac{5}{3}\ln\mu^2l^2) \end{aligned} \tag{4.7}$$

It is interesting to notice that the result of Bern *et al.* (1978*a*) is valid for any regulators $V(-p^2l^2)$ if, in their final expressions for loop diagrams, the coefficients $\frac{1}{3}\Lambda^2$ and $\ln(\Lambda^2/\mu^2)$ should be changed to σl^{-2} and $-\ln(\mu^2l^2)$, respectively.

The attraction of our approach is that the nonlocal scheme is unitary in the presence of the analytic regulator [for details, see Efimov (1985)]. In our case, supplementary singularities caused by regulators do not exist and the analytic properties of any diagrams are conserved at a finite value of momentum variables p^2. In contrast, for meromorphic regulators like the Pauli–Villars regularization procedure, the analytic properties of diagrams

Table II

Diagram	Leading terms in sum of counterterm diagrams
Fig. 8a′	$-[\delta m^2+(Z_\phi-1)(p^2+m^2)](p^2+m^2)^{-2}$
Fig. 8b′	$-g(Z_g-1)[(m^2+p_1^2)(m^2+p_2^2)(m^2+p_3^2)]^{-1}$
Fig. 8c′	$g[(p_1^2+m^2)(p_2^2+m^2)(p_3^2+m^2)]^{-1}$ $\times\left[\frac{\delta m^2+(Z_\phi-1)(p_1^2+m^2)}{p_1^2+m^2}+\frac{\delta m^2+(Z_\phi-1)(p_2^2+m^2)}{p_2^2+m^2}+\frac{\delta m^2+(Z_\phi-1)(p_3^2+m^2)}{p_3^2+m^2}\right]$

are broken, which leads to difficulties in the proof of analyticity and the unitarity of the regularized theory with these types of regulators. In the last case, one expects that unitarity is regained as the regularization is removed $\Lambda \to \infty$, by which, of course, singularities (poles) are displaced at infinity.

5. NONLOCAL STOCHASTIC QUANTIZATION OF GAUGE FIELDS

At first sight, the majority of physicists seem to think that the stochastic quantization method is little more than an amusing alternative to conventional Hamiltonian, path integral, and action formulations. It turns out that this method has given birth to a number of new ideas and is very useful for understanding many problems of field theory in light of its present developments. As mentioned by Bern *et al.* (1987*b*), these developments are Zwanziger gauge fixing (Zwanziger, 1981; Floratos *et al.*, 1984), large-N quenching and large-N master fields (Alfaro and Sakita, 1983; Greensite and Halpern, 1983), stochastic stabilization (Greensite and Halpern, 1984), stochastic regularization (Bern *et al.*, 1987*b*; Niemi and Wijewardhana, 1982; Breit *et al.*, 1984; Namiki and Yamanaka, 1984; Bern, 1985), the QCD_4 maps which run in ordinary time (Claudson and Halpern, 1985; Bern and Chan, 1986), and numerical applications of the Langevin equation in lattice gauge theory (Hamber and Heller, 1984; Batrouni *et al.*, 1985). For review see Namsrai (1986) and Migdal (1986), where earlier references concerning this problem are cited.

To introduce nonlocality into the stochastic quantization formalism for gauge fields, we follow Bern *et al.* (1987*b*). Our procedure is very similar to theirs. However, our method is more general and deals with any form factor of the type $V(-p^2l^2)$.

5.1. Nonlocal Langevin Systems for Gauge Theory

The nonlocal Parisi-Wu Langevin system for $SU(N)$ Yang-Mills theory in d dimensions is given by

$$\dot{A}^a_\mu(x, t) = -\frac{\delta S}{\delta A^a_\mu}(x, t) + d^{ab}_\mu Z^b(x, t) + \int (dy)\, K^{ab}_{xy}(\Delta)\eta^b_\mu(y, t) \quad (5.1)$$

where local noise satisfies the relation

$$\langle \eta^a_\mu(x, t)\eta^b_\nu(y, t')\rangle_\eta = 2\delta^{ab}\delta_{\mu\nu}\delta(t-t')\delta^d(x-y) \quad (5.2)$$

and $K^{ab}_{xy}(\Delta)$ is nonlocal distribution discussed in previous sections. According to the equilibrium hypothesis, the nonlocal Euclidean Green functions determined by vacuum expectation values of products of fields

$$\langle F[A(\cdot)]\rangle_0 = \langle A_{\nu_1}(x_1) \cdots A_{\nu_n}(x_n)\rangle_0 = \prod_{i\neq j} D_{\nu_i\nu_j}(x_i - x_j) \tag{5.3}$$

in the usual nonlocal quantum field theory (for example, Efimov, 1985; Namsrai, 1986) are now given by

$$\langle F[A(\cdot)]\rangle \equiv \lim_{t\to\infty} \langle F[A(\cdot, t)]\rangle_\eta \tag{5.4}$$

where $F[A]$ is any equal fifth-time functional (product) of the gauge field $A_\mu^a(\eta)$. In particular, the nonlocal propagator for the photon field $A_\mu(x)$ in (5.3) takes the form

$$D_{\mu\nu}(x-y) = \langle 0|T(A_\mu(x)A_\nu(y))|0\rangle = \frac{ig_{\mu\nu}}{(2\pi)^4}\int d^4p\, e^{-ip(x-y)}\frac{V(-p^2l^2)}{p^2}$$

in accordance with the nonlocal theory. Here, the form factor $V(-p^2l^2)$ is given by formula (2.9b) with $m=0$.

Our notation in (5.1) is

$$S = \tfrac{1}{4}\int (dx)\, F_{\mu\nu}^a(x)F_{\mu\nu}^a(x)$$

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c$$

In this paper we use the following covariant derivative:

$$d_\mu^{ab} = \delta^{ab}\partial_\mu + gf^{abc}A_\mu^c, \qquad (dx)\equiv d^dx$$

In (5.1) we have chosen to add a Zwanziger gauge-fixing term $d_\mu^{ab}Z^b$, which we will specify as $\alpha Z^a = \partial A^a$ for computational purposes. As shown below, gauge-invariant quantities do not depend on the gauge fixing for the nonlocal case. The nonlocal distribution $K_{xy}^{ab}(\Delta)$ is a function of the covariant Laplacian

$$\begin{aligned}\Delta_{xy}^{ab} &\equiv \int (dz)\,(d_\mu)_{xz}^{ac}\,(d_\mu)_{zy}^{cb}\\ (d_\mu)_{xy}^{ab} &= d_\mu^{ab}(x)\,\delta^d(x-y)\end{aligned} \tag{5.5}$$

so that

$$K_{yx}^{ba}(\Delta) = K_{xy}^{ab}(\Delta)$$

In the weak coupling limit the Langevin equation (5.1) has the equivalent integral formulation

$$A_\mu^a(x, t) = \int_{-\infty}^{t} dt' \, (dy) \, G_{\mu\nu}^{ab}(x-y, t-t') \times \left[W_\nu^b(y, t') + \frac{1}{\alpha} Y_\nu^b(y, t') + \int (dz) \, K_{yz}^{bc}(\Delta) \eta_\nu^c(z, t') \right] \tag{5.6}$$

where

$$G_{\mu\nu}^{ab}(x-y, t-t') = \delta^{ab} \Theta(t-t') \int (dp) \, e^{-ip(x-y)} \times [T_{\mu\nu}(p) \, e^{-p^2(t-t')} + L_{\mu\nu}(p) \, e^{-p^2(t-t')/\alpha}] \tag{5.7}$$

is the Langevin Green function, which is determined by usual procedure:

$$G_{\mu\nu}^{ab}(x, t) = \delta^{ab}[T_{\mu\nu} G^T(x, t) + L_{\mu\nu} G^L(x, t)]$$

Here $T_{\mu\nu}$ and $L_{\mu\nu}$ are the standard transverse and longitudinal projection operators, respectively. In the momentum space they take the form

$$T_{\mu\nu}(k) = \delta_{\mu\nu} - k_\mu k_\nu / k^2$$

$$L_{\mu\nu}(k) = k_\mu k_\nu / k^2$$

In (5.6) we have defined the interaction terms

$$W_\nu^b = -g f^{bcd} [\partial_\beta (A_\beta^c A_\nu^d) - (\partial_\beta A_\nu^c) A_\beta^d + (\partial_\nu A_\beta^c) A_\beta^d] + g^2 f^{bdc} f^{cne} A_\beta^n A_\nu^e A_\beta^d \tag{5.8}$$

$$Y_\nu^b \equiv g f^{bcd} A_\nu^d (\partial A^c) \tag{5.9}$$

The former arises from the action and last term is due to the Zwanziger one. In expression (5.6) we have also employed the technical device of choosing $t_0 = -\infty$, so that the system has equilibrated at any finite fifth time.

A method of form-factor expansion in powers of the coupling constant plays an important role in the proof of gauge invariance of the nonlocal stochastic quantization theory. As a first step in this expansion, we write, in accordance with Bern *et al.* (1987*b*),

$$\Delta_{xy}^{ab} = \delta^{ab} \Box_{xy} l^2 + g(\Gamma_1)_{xy}^{ab} + g^2 (\Gamma_2)_{xy}^{ab} \tag{5.10}$$

where the regulator "vertices" Γ_1 and Γ_2 are defined as

$$(\Gamma_1)_{xy}^{ab} = l^2 f^{abc} (\partial_\mu^x A_\mu^c(x) + A_\mu^c(x) \partial_\mu^x) \delta^d(x-y) \tag{5.11a}$$

$$(\Gamma_2)_{xy}^{ab} = l^2 f^{amn} f^{nbe} A_\mu^m(x) A_\mu^e(x) \, \delta^d(x-y) \tag{5.11b}$$

In (5.11) the derivatives ∂_μ^x act on everything to the right. Further, for any distribution of the type of (2.5) we may write down the following expansion rule:

$$
\begin{aligned}
K_{xy}^{ab}(\Delta) &= \sum_{n=0}^{\infty} \frac{c_n}{(2n)!} (\Delta_{xy}^{ab})^n \\
&= \sum_{n=0}^{\infty} \frac{c_n}{(2n)!} \delta^{ab} l^{2n} \Box_{xy}^n \\
&\quad + \sum_{n=0}^{\infty} \frac{c_n l^{2n-2}}{(2n)!} \int (dz)[g(\Gamma_1)_{xz}^{ab} + g^2(\Gamma_2)_{xz}^{ab}] \Box_{zy}^{n-1} n \\
&\quad + \tfrac{1}{2} \sum_{n=0}^{\infty} \frac{c_n}{(2n)!} l^{2n-4} \int (dz_1)(dz_2) g^2 (\Gamma_1)_{xz_1}^{ac} (\Gamma)_{z_1 z_2}^{cb} \Box_{z_2 y}^{n-2} n(n-1) \\
&\quad + \cdots \qquad (5.12) \\
&= \delta^{ab} K_{xy}(\Box) + \frac{g}{2} \int (dz_1)\,(dz_2) \\
&\quad \times [K_{xz_1}^{(1)}(\Box)(\Gamma_1)_{z_1 z_2}^{ab} H_{z_2 y}(\Box) \\
&\quad + H_{xz_1}(\Box)(\Gamma_1)_{z_1 z_2}^{ab} K_{z_2 y}^{(1)}(\Box)] + \frac{g^2}{2} \int (dz_1)\,(dz_2) \\
&\quad \times [K_{xz_1}^{(1)}(\Box)(\Gamma_2)_{z_1 z_2}^{ab} H_{z_2 y}(\Box) + H_{xz_1}(\Box)(\Gamma_2)_{z_1 z_2}^{ab} K_{z_2 y}^{(1)}(\Box)] \\
&\quad + \frac{g^2}{6} \int (dz_1) \cdots (dz_4)[K_{xz_1}^{(2)}(\Box)(\Gamma_1)_{z_1 z_2}^{ac} H_{z_2 z_3}(\Box)(\Gamma_1)_{z_3 z_4}^{cb} H_{z_4 y}(\Box) \\
&\quad + H_{xz_1}(\Box)(\Gamma_1)_{z_1 z_2}^{ac} K_{z_2 z_3}^{(2)}(\Box)(\Gamma_1)_{z_3 z_4}^{cb} H_{z_4 y}(\Box) \\
&\quad + H_{xz_1}(\Box)(\Gamma_1)_{z_1 z_2}^{ac} H_{z_2 z_3}(\Box)(\Gamma_1)_{z_3 z_4}^{cb} H_{z_4 y}^{(2)}(\Box)] + \cdots
\end{aligned}
$$

Here the Fourier transforms of generalized functions $K^i(\Box)$ are given by

$$
\begin{aligned}
K(-p^2 l^2) &= \frac{1}{2i} \int_{-\beta+i\infty}^{-\beta-i\infty} d\xi \frac{w(\xi)}{\sin \pi\xi} (p^2 l^2)^\xi \\
K^{(1)}(-p^2 l^2) &= \frac{1}{2i} \int_{-\beta+i\infty}^{-\beta-i\infty} d\xi \frac{w(\xi)}{\sin \pi\xi} \xi (p^2 l^2)^\xi \qquad (5.3) \\
K^{(2)}(-p^2 l^2) &= \frac{1}{2i} \int_{-\beta+i\infty}^{-\beta-i\infty} d\xi \frac{w(\xi)}{\sin \pi\xi} \xi(\xi-1)(l^2 p^2)^\xi
\end{aligned}
$$

and for the operator $H_{xy}(\Box) = (\Box_x l^2)^{-1} \delta^d(x-y)$ we have

$$
\begin{aligned}
H_{xy}(\Box) &= -\int (dp) H(p^2 l^2)\, e^{-ip(x-y)} \\
H(p^2 l^2) &= \frac{1}{p^2 l^2}
\end{aligned}
\qquad (5.14)
$$

With the form-factor expansion (5.12) for any desired order it is not difficult to iterate the integral equation (5.6) for the Langevin field

$$A_\mu[\eta] = \sum_{m=0}^{\infty} g^m A_\mu^{(m)}[\eta] \tag{5.15}$$

up to arbitrarily high order as well. As an example, the result for the form factor $K(\Delta)$ in $d=4$ dimensions takes the form

$$A_\mu^{(0)a}(x,t) = \int_{-\infty}^{t} dt'\,(dy)\, G_{\mu\nu}^{ab}(x-y,t-t') \int (dz)\, K_{zy}(\Box)\eta_\nu^b(z,t') \tag{5.16a}$$

$$\begin{aligned} A_\mu^{(1)a}(x,t) = \int_{-\infty}^{t} dt'\,(dy)\, G_{\mu\nu}^{ab}(x-y,t-t') \\ \times\Big\{ W_\nu^{(0)b}(y,t') + \frac{1}{\alpha} Y_\nu^{(0)b}(y,t') + \tfrac{1}{2}\int (dz)[K^{(1)}(\Box)\Gamma_1(A^0)H(\Box) \\ + H(\Box)\Gamma_1(A^0)K^{(1)}(\Box)]_{yz}^{bc}\eta_\nu^c(z,t')\Big\} \end{aligned} \tag{5.16b}$$

$$\begin{aligned} A_\mu^{(2)a}(x,t) = \int_{-\infty}^{t} dt'\,(dy)\, G_{\mu\nu}^{ab}(x-y,t-t') \\ \times\Big\{ W_\nu^{(1)b}(y,t') + \frac{1}{\alpha} Y_\nu^{(1)b}(y,t') + \int (dz) \\ \times\{\tfrac{1}{2}[K^{(1)}(\Box)\Gamma_1(A^{(1)})H(\Box) + H(\Box)\Gamma_1(A^{(1)})K^{(1)}(\Box)] \\ + \tfrac{1}{2}[K^{(1)}(\Box)\Gamma_2(A^{(0)})H(\Box) + H(\Box)\Gamma_2(A^{(0)})K^{(1)}(\Box)] \\ + \tfrac{1}{6}[K^{(2)}(\Box)\Gamma_1(A^{(0)})H(\Box)\Gamma_1(A^{(0)})H(\Box) \\ + H(\Box)\Gamma_1(A^{(0)})K^{(2)}(\Box)\Gamma_1(A^{(0)})H(\Box) \\ + H(\Box)\Gamma_1(A^{(0)})H(\Box)\Gamma_1(A^{(0)})K^{(2)}(\Box)]\}_{yz}^{bc}\eta_\nu^c(z,t')\Big\} \end{aligned} \tag{5.16c}$$

Here, the product of the operators in (5.16b) and (5.16c) should be understand as a contraction operation between them [see formula (5.12)].

5.2. Langevin Tree Graphs

We note that more useful at arbitrary order is the equivalent description in terms of Langevin tree graphs, which are easily derived from equation (5.6) or (5.16). For this purpose, the tree-graph expansions of the form factor should be given, as done by Bern *et al.* (1987*b*) for the concrete regulator $[R(\Delta)]_{xy}^{ab} = \delta^{ab}[1-\Delta/l^2]_{xy}^{-1}$. In the nonlocal theory the Langevin tree graphs through $O(g^2)$ are shown in Figure 10. These diagrams may be constructed to all orders using the Langevin tree rules given in Figure 9.

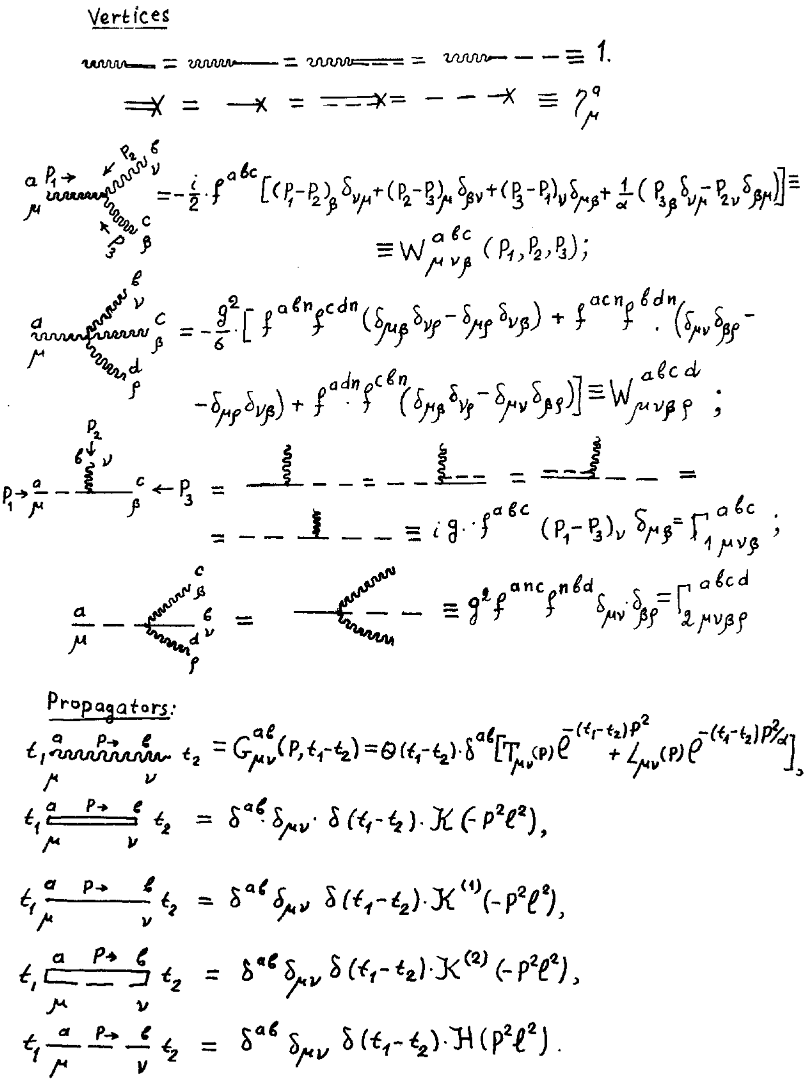

Fig. 9. Langevin tree rules using nonlocal form factors.

As a trivial example, we obtain the zeroth-order two-point function. From the solution (5.16a) it follows in accordance with the local noise property (5.2)

$$\langle A_\mu^{(0)a}(x,t) A_\nu^{(0)b}(y,t)\rangle = \delta^{ab} \int (dp)\, e^{-ip(x-y)}\, [T_{\mu\nu}(p) + \alpha L_{\mu\nu}(p)]\, \frac{V(-p^2 l^2)}{p^2} \tag{5.17}$$

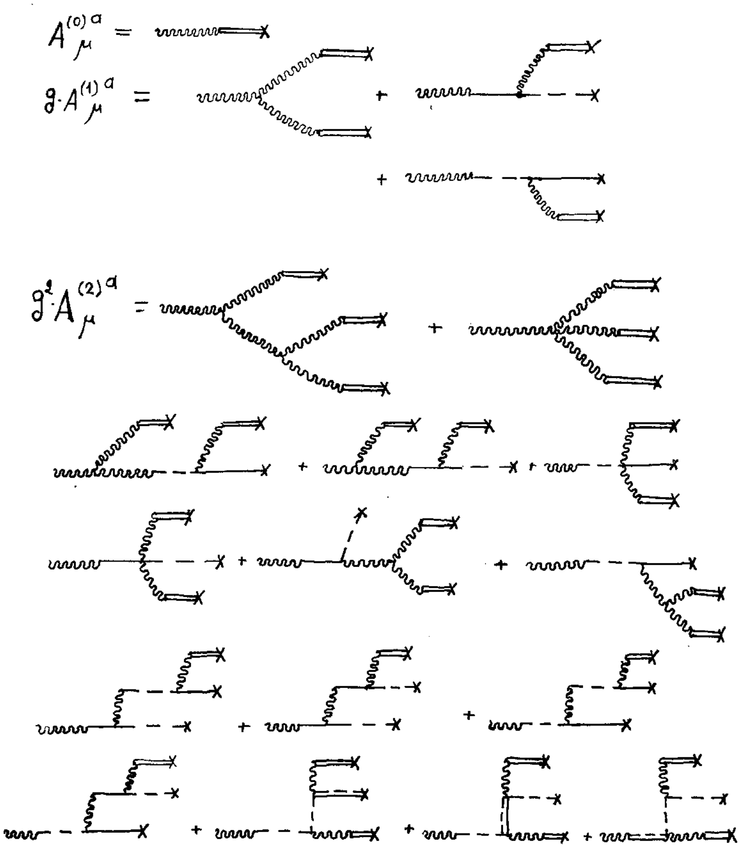

Fig. 10. Langevin tree diagrams through $O(g^2)$ in the nonlocal stochastic scheme.

or, using the Langevin tree diagram shown in Figure 11,

$$
\begin{aligned}
&D^{ab}_{\mu\nu}(t_1, t_2; p)\\
&\quad \equiv 2\int_{-\infty}^{t_1} dt_3 \int_{-\infty}^{t_2} dt_4\, G^{ac}_{\mu\rho}(p, t_1-t_3) G^{bc}_{\nu\rho}(p, t_2-t_4)\delta(t_3-t_4)K^2(-p^2l^2)\\
&\quad = \delta^{ab}[T_{\mu\nu}(p)\, e^{-p^2|t_1-t_2|} + \alpha L_{\mu\nu}(p)\, e^{-p^2|t_1-t_2|/\alpha}]\,\frac{V(-p^2l^2)}{p^2}
\end{aligned}
$$

The result for the nonlocal free gluon propagator is just (5.17). Other free

Fig. 11. A simple contraction for the nonlocal theory with form factor $V(-p^2l^2)$.

nonlocal Green functions are constructed according to the usual Wick expansion in terms of the result (5.17).

In the next section, we apply these Langevin equations and their rules for the nonlocal stochastic quantization theory to the computation of the one-loop gluon mass.

6. VANISHING GLUON MASS IN THE NONLOCAL STOCHASTIC QUANTIZATION THEORY

Verification of gauge invariance in the nonlocal stochastic quantization scheme with arbitrary form factors is crucial for its further development. We will verify in this section that the QCD_4 gluon mass remains zero at the one-loop level, with any form factors $V(-p^2l^2)$ or $K(-p^2l^2)$. Our approach to this problem is as follows. First, we construct expressions

$$\Pi^{ab}_{\mu\nu}(x-y)=\langle A^{(1)a}_{\mu}(x,t)A^{(1)b}_{\nu}(y,t)\rangle_{\eta}$$

and

$$N^{ab}_{\mu\nu}(x-y)=\langle A^{(2)a}_{\mu}(x,t)A^{(0)b}_{\nu}(y,t)\rangle_{\eta}$$

by using equations (5.16). Second, with these formulas, we sketch corresponding diagrams. It turns out that there are 47 distinct Langevin graphs in the two-point function at order g^2, where diagrams trivially related by symmetry are not included in the count. As a particular case of Bern *et al.* (1987*b*), it is seen that only ten make nonzero contributions to the mass renormalization, while only two contribute to the wave function and gauge parameter (α) renormalizations.

Following Bern *et al.* (1987*b*), we have found it convenient to group the 47 diagrams into four classes (see diagrams sketched in Figures 12–15) of which only the first class contributes to the wave function and α renormalizations, and only the first two classes contribute to the mass renormalization. The third class contributes only to the finite part of the vacuum polarization, which will not be considered in this paper, while the diagrams in the fourth class vanish identically.

The structure of the diagrams shown in Figures 12–15 is similar to those considered by Bern *et al.* (1987*b*). Therefore, we do not discuss them in detail and indicate only some of their peculiarities. For example, the diagrams shown in Figure 12 contain only (Zwanziger gauge-fixed) Yang–Mills vertices and no form-factor vertices, while the diagrams in Figure 13 contain at least one Γ_1 or Γ_2 regulator vertex, and provide the additional gluon mass contributions needed to cancel the contribution of the ordinary graphs (Figure 12). For this class of diagrams, contributions to wave function or α renormalizations are absent. The diagrams shown in Figure 14, also contain regulator vertices, but contribute only to the finite part of the vacuum

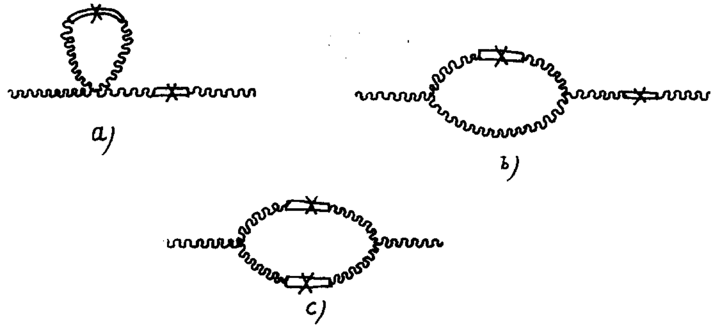
a) b) c)

Fig. 12. "Ordinary" nonvanishing Langevin diagrams in the nonlocal stochastic quantization scheme.

polarization. Finally, the group of diagrams in Figure 15 vanishes identically. Some (the tadpole loops) vanish as usual by f^{abc} antisymmetry. The remaining diagrams vanish due to the (fifth-time) retarded property of the Langevin Green functions, which contribute a factor of $\theta(t_1-t_2)\theta(t_2-t_1)=0$ to each diagram.

In order to compute explicit contributions to renormalization of the mass correction due to the diagrams shown in Figures 12 and 13, we study expressions $\Pi^{ab}_{\mu\nu}(x-y)=\langle A^{(1)a}_{\mu}(x,t)A^{(1)b}_{\nu}(y,t)\rangle_{\eta}$. Thus, taking into account the formula (5.16b), it is easily seen that explicit contribution from the

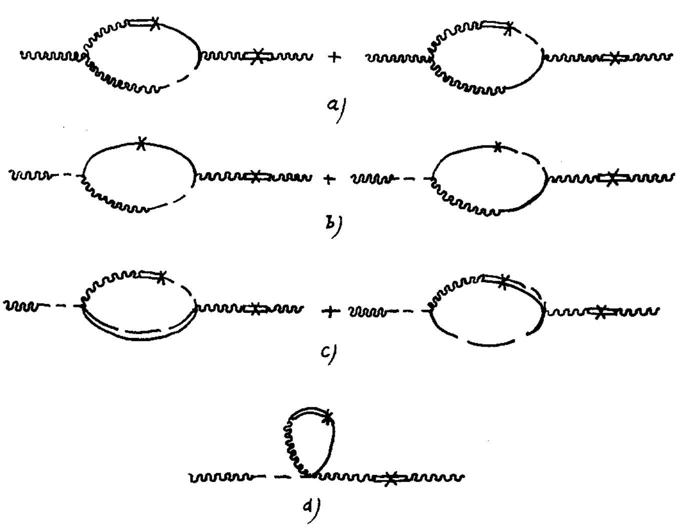
a) b) c) d)

Fig. 13. Diagrams with nonlocal regulator vertices that also contribute to gluon mass.

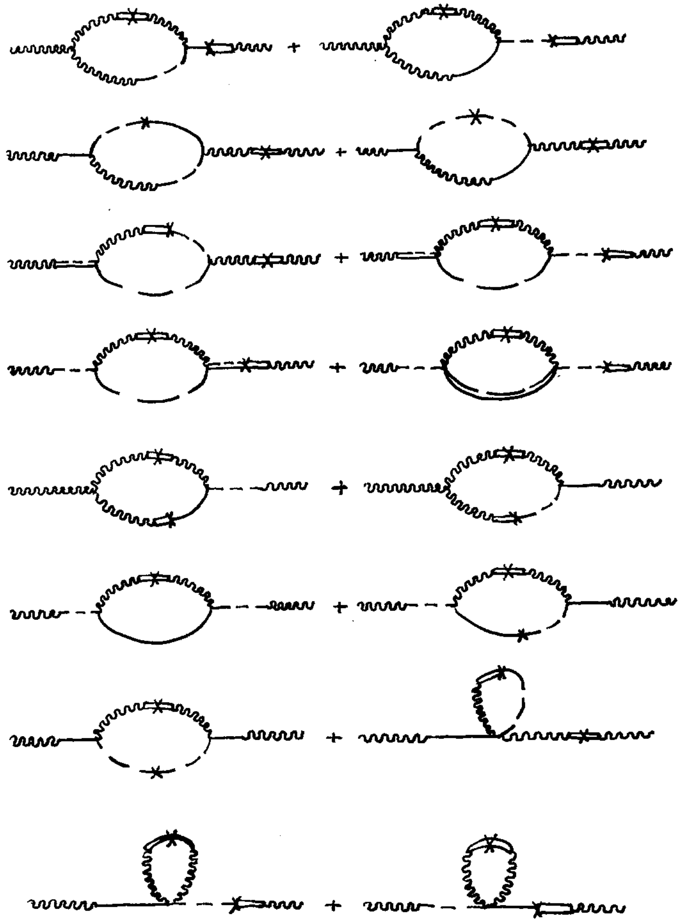

Fig. 14. Diagrams with nonlocal regulator vertices, which are finite as $l \to 0$.

diagram of Figure 12c is calculated by using the formula

$$\Pi^{ab}_{\mu\nu}(x-y) = \langle \Sigma^a_\mu(x, t)\Sigma^b_\nu(-y, t)\rangle_\eta \tag{6.1}$$

where

$$\Sigma^a_\mu(x, t) = \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt_1 \int_{-\infty}^{t'} dt_2 \int (dp_1)\,(dp_2)\,(dp_3)\, e^{-ip_1 x}\, G^{aa_1}_{\mu\mu_1}(p_1, t-t')$$
$$\times G^{en}_{\beta\delta}(p_2, t'-t_1) K(-p_2^2 l^2)\bar{\delta}^d(p_1+p_2+p_3)$$
$$\times \{W^{a_1 ec}_{\mu_1\beta\theta}(p_1, p_2, p_3) G^{cs}_{\theta\rho}(p_3, t'-t_2)$$
$$\times K(-p_3^2 l^2) - \tfrac{1}{2}\Gamma^{a_1 es}_{1\mu_1\rho\beta}\delta(t'-t_2)[H(p_1^2 l^2)K^{(1)}(-p_3^2 l^2)$$
$$+ H(p_3^2 l^2)K^{(1)}(-p_1^2 l^2)]\}\eta^n_\delta(p_2, t_1)\eta^s_\rho(p_3, t_2)$$

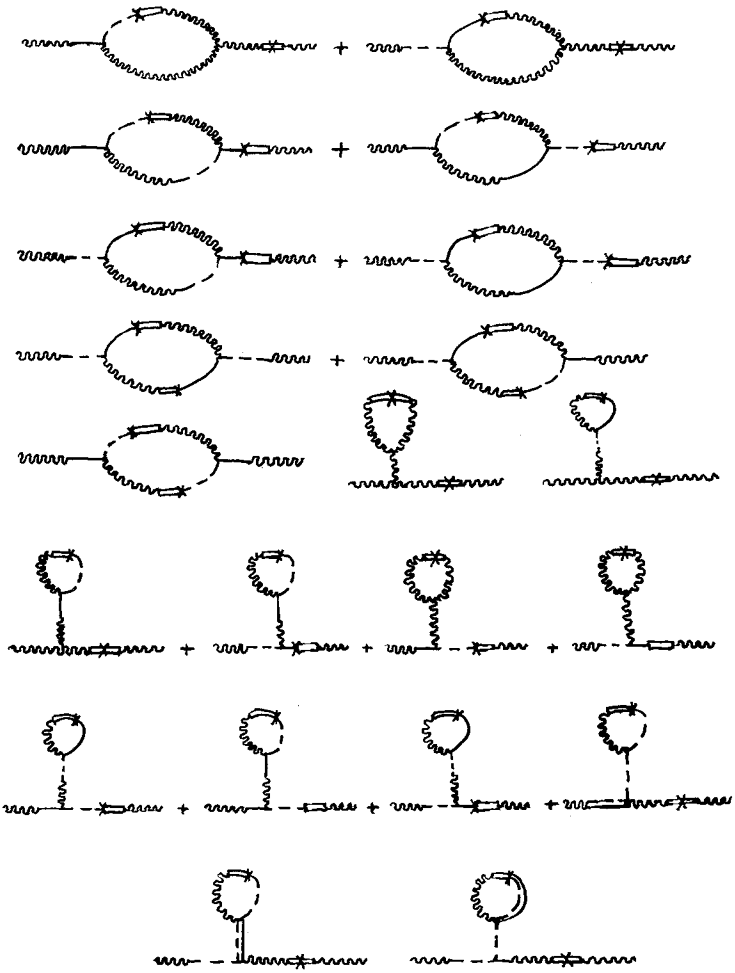

Fig. 15. Diagrams that vanish identically in the nonlocal stochastic scheme.

Here the explicit form of the vertices $W^{a_1ec}_{\mu_1\beta\theta}(p_1, p_2, p_3)$ and $\Gamma^{a_1es}_{1\mu_1\rho\beta}$ is sketched in Figure 9. The majority of terms in (6.1) correspond to some finite and zero diagrams in Figures 14 and 15. Further, according to the formula (5.2), we make a noise contraction in (6.1), perform the fifth-time integrations, separate terms giving contributions in accordance with the diagram of Figure 12c, and integrate over the momentum variable with form factor $V(-p^2l^2)$. Thus, after some tensor algebra, we obtain the explicit leading value for this diagram near $p=0$ as

$$\Pi^{ab}_{1\mu\nu}(p) = -f^{amn} f^{bmn} \frac{g^2}{16\pi^2} \Delta_{\mu\beta}(p)(\tfrac{5}{2}\delta_{\beta\nu}p^2) \ln l^2\mu^2$$

where

$$\Delta_{\mu\beta}(p) = [T_{\mu\beta}(p) + \alpha L_{\mu\beta}(p)] p^{-2}$$

Truncation near $p=0$ is accomplished by removal of the factor $\Delta_{\mu\nu}(p)$. We see that this term contributes to the wave function renormalization only. Now we study diagrams which give contribution to the gluon mass renormalization.

Contributions to the mass renormalization due to the diagram of Figure 12a arise from the contraction result between the second term of (5.16c) and $A^{(0)b}_\nu$ in (5.16a):

$$\begin{aligned}\Pi^{ab}_{2\mu\nu}(p) = \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt_1 \int_{-\infty}^{t'} d\tau\, (dp_2)\, G^{aa_1}_{\mu\beta}(p, t-t')\Theta(t'-t_1)\Theta(t'-\tau) \\ \times \{\delta^{km}\delta^{eb}\Delta'_{\rho_1\rho_2}(p_2, t'-t_1)\Delta'_{\rho_3\nu}(p, t'-\tau) \\ + \delta^{me}\delta^{kb}\Delta'_{\rho_1\rho_3}(p_2, t'-t_1)\Delta'_{\rho_2\nu}(p, t'-\tau) \\ + \delta^{mb}\delta^{ke}\Delta'_{\rho_1\nu}(p, t'-\tau)\Delta'_{\rho_2\rho_3}(p_2, t'-t_2)\} \\ \times V(-p_2^2 l^2)\, V(-p^2 l^2)\, W^{a_1 mke}_{\beta\rho_1\rho_2\rho_3}\end{aligned}$$

where

$$\Delta'_{\rho_1\rho_2}(p, t) = T_{\rho_1\rho_2}(p)\, e^{-tp^2} + L_{\rho_1\rho_2}(p)\, e^{-tp^2/\alpha}$$

and $W^{a_1 mke}_{\beta\rho_1\rho_2\rho_3}$ is presented in Figure 9. After an elementary calculation, we get

$$\Pi^{ab}_{2\mu\nu}(p) = -f^{amn} f^{bmn} g^2 \Delta_{\mu\beta}(p)\Delta_{\beta\nu}(p) \frac{3+\alpha}{4} 3 \int (dq) \frac{V(-q^2 l^2)}{q^2}$$

The infraviolet divergence in this term is caused by the zero mass of the gluon field. Assuming $q^2 \to q^2 + \varepsilon$, we obtain

$$\Pi^{ab}_{2\mu\nu}(p) = f^{amn} f^{bmn} g^2 \Delta_{\mu\beta}(p)\Delta_{\beta\nu}(p) \frac{1}{16\pi^2}\left(-\frac{3\sigma}{l^2}\frac{3+\alpha}{4}\right) \qquad (6.2)$$

Here for $SU(N)$, $f^{amn} f^{bmn} = \delta^{ab} N$.

Now we calculate corrections to the gluon mass renormalization due to the diagrams shown in Figures 12b and 13a, which are calculated by using the contraction of first term in (5.16c) with $A_\nu^{(0)b}(x, t)$. The corresponding expressions take the form

$$\Pi_{3\mu\sigma}^{as}(p) = 16 \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt_1 \, (dp_1) \, V(-p^2 l^2) V(-(p-p_1)^2 l^2)$$
$$\times \Theta(t-t')\Theta(t'-t_1)\Theta(t-t_1)\Delta'_{\nu_1\nu_2}(p_1, t'-t_1)$$
$$\times \Delta'_{\mu\nu}(p, t-t')\Delta'_{\lambda_1\sigma}(p_1, t-t_1)$$
$$\times \Delta'_{\beta_1\rho_1}(p-p_1, t'-t_1) \, W^{anm}_{\nu\beta_1\nu_1}(-p, p-p_1, p_1)$$
$$\times W^{mns}_{\nu_2\rho_1\lambda_1}(-p_1, p_1-p, p) \tag{6.3}$$

and

$$\Pi_{4\mu\sigma}^{as}(p) = 2g^2\delta^{as}Nl^2 \int (dq) \left\{ \frac{q_\nu q_\sigma}{2q^4}(\alpha-3) V(-q^2 l^2) K(-p^2 l^2) \right.$$
$$\times [H(q^2 l^2) K^{(1)}(-p^2 l^2) + H(p^2 l^2) K^{(1)}(-q^2 l^2)]$$
$$+ \frac{4q_\nu q_{\lambda_1}}{q^2} V(-p^2 l^2) K(-q^2 l^2)$$
$$\left. \times H(q^2 l^2) K^{(1)}(-q^2 l^2) \right\} \Delta_{\sigma\lambda_1}(p)\Delta_{\mu\nu}(p) \tag{6.4}$$

respectively. In (6.3) integration over fifth-time variables should be carried out, after which this expression is reduced to the analogous formula for $\Pi_{4\mu\nu}^{as}(p)$ in (6.4):

$$\Pi_{3\mu\sigma}^{as}(p) = \delta^{ab} N g^2 \Delta_{\mu\nu}(p) \Delta_{\lambda_1\sigma}(p) \left[\frac{5}{4} + \frac{3\alpha}{4} \right] \delta_{\nu\lambda_1} \int (dq) \frac{V(-q^2 l^2)}{q^2}$$
$$= \delta^{ab} N g^2 \Delta_{\mu\nu}(p) \Delta_{\nu\sigma}(p) \left[\frac{5}{4} + \frac{3\alpha}{4} \right] \frac{\sigma}{l^2} \frac{1}{16\pi^2} \tag{6.5}$$

By definition (5.13) for the form factor $K^i(-p^2 l^2)$, it is easily seen that first term with $K^{(1)}(-p^2 l^2)$ in (6.4) goes to zero at the limit $p^2 \to 0$ and the main asymptotic of its second term is constant, so that third term gives the following leading term:

$$\Pi_{4\mu\sigma}^{as}(p) = \frac{\delta^{as} N g^2}{16\pi^2} \Delta_{\mu\nu}(p) \Delta_{\nu\sigma}(p) \left[-\frac{\sigma}{l^2} \right] \tag{6.6}$$

Analogously, contributions to the mass renormalization in QCD_4 due to the diagrams shown in Figures 13b-13d are calculated by using contraction of third, fourth, and fifth terms in (5.16c) with $A_\nu^{(0)b}(x, t)$. The corresponding result reads

$$\Pi^{as}_{5\mu\sigma}(p) = \frac{\delta^{as}Ng^2}{16\pi^2}\Delta_{\mu\nu}(p)\Delta_{\nu\sigma}(p)V(-p^2l^2)H(p^2l^2)$$

$$\times\frac{1}{l^2}\left[\frac{w(-2)}{2} - \frac{w(-2)}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} d\xi \frac{w(\xi)}{\sin\pi\xi}\frac{\Gamma(-\xi)}{\Gamma(2-\xi)}\varepsilon^{\xi}l^{2\xi}\right] \quad (6.7a)$$

$$\Pi^{as}_{6\mu\sigma}(p) = -\frac{\delta^{as}Ng^2}{16\pi^2}\Delta_{\mu\nu}(p)\Delta_{\nu\sigma}(p)\left[\frac{w(-2)}{2l^2}\right]V(-p^2l^2)H(p^2l^2) \quad (6.7b)$$

$$\Pi^{as}_{7\mu\sigma}(p) = \frac{\delta^{as}Ng^2}{16\pi^2}\Delta_{\mu\nu}(p)\Delta_{\nu\sigma}(p)V(-p^2l^2)H(p^2l^2)\frac{w(-2)}{l^2} \quad (6.7c)$$

$$\times\frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} d\xi \frac{w(\xi)}{n\pi\xi}\frac{\Gamma(-\xi)}{\Gamma(2-\xi)}\varepsilon^{\xi}l^{2\xi}$$

In expressions (6.2) and (6.5)-(6.7), truncation near $p = 0$ is accomplished by removal of the two factors $\Delta_{\alpha\beta}(p)$; all resulting sums in the contributions of these diagram are zero, so the gluon remains massless in this order for the nonlocal stochastic quantization theory with arbitrary form factors. This generalizes the regularized scheme proposed by Bern *et al.* (1987*b*).

Thus, the nonlocal method presented here for the Langevin and Schwinger-Dyson formalisms of stochastic quantization gives ultraviolet finiteness to all orders for gauge theory Green functions in d dimensions and ensures its gauge invariance. The latter is achieved by using the covariant Laplacian function (in which the gauge-fixing term is absent) in the construction of the theory. In our case, the nonlocal distribution $K_{xy}(\Box)$ is translation invariant and so a gauge-covariant parallel transport of the local noise guarantees the gauge covariance of the regularized Langevin system under the local d-dimensional gauge transformation [for details, see Bern *et al.* (1987*b*)]:

$$\dot{A}^a_\mu(x, t) \Rightarrow \Omega^{ab}(x)\dot{A}^b_\mu(x, t)$$

$$\eta^a_\mu(x, t) \Rightarrow \Omega^{ab}(x)\eta^b_\mu(x, t)$$

$$K^{ab}_{xy}(\Delta) \Rightarrow \Omega^{aa'}(x)\Omega^{bb'}(y)K^{a'b'}_{xy}(\Delta)$$

where $\Omega(x) \in SO(N^2 - 1)$ is in the adjoint representation of $SU(N)$.

7. SCALAR ELECTRODYNAMICS

For concrete computational purposes, we present here the construction method of the electrodynamics of charged spinless particles and illustrate the extension of the scheme to include matter multiplets. As in Yang-Mills, the basic idea is that gauge invariance is maintained by choosing each form factor as a function of the covariant derivative in the relevant representation.

The nonlocal and Zwanziger gauge-fixed Langevin system for scalar electrodynamics (SED) takes the form

$$\dot{A}_\mu(x, t) = -\frac{\delta S}{\delta A_\mu}(x, t) + \partial_\mu Z(x, t) + \int (dy)\, K_{xy}(\Box)\eta_\mu(y, t) \tag{7.1a}$$

$$\dot{\phi}(x, t) = -\frac{\delta S}{\delta \phi^*}(x, t) + ie\phi(x, t)Z(x, t) + \int (dy) K_{xy}(\Delta)\eta(y, t) \tag{7.1b}$$

$$\dot{\phi}^*(x, t) = -\frac{\delta S}{\delta \phi}(x, t) - ie\phi^*(x, t)Z(x, t) + \int (dy)\, K_{xy}(\Delta^*)\eta^*(y, t) \tag{7.1c}$$

where local noises satisfy the usual relations

$$\langle \eta_\mu(x, t)\eta_\nu(y, t')\rangle = 2\delta_{\mu\nu}\delta(t-t')\delta^d(x-y) \tag{7.2a}$$

$$\langle \eta^*(x, t)\eta(y, t')\rangle = 2\delta(t-t')\delta^d(x-y) \tag{7.2b}$$

Here

$$S = \int (dx)\, [\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + |(\partial_\mu - ieA_\mu)\phi|^2] \tag{7.2c}$$

is the usual Euclidean action of SED constructed by using local fields $A_\mu(x, t)$ and $\phi(x, t)$. In contrast to nonlocal quantum field theory (Efimov, 1977, 1985), the interaction Lagrangian in (7.2c) is local. The appropriate covariant Laplacians for the charged scalar fields are

$$\begin{aligned}
\Delta_{xy} &= \int (dz)\, (D_\mu)_{xz}(D_\mu)_{zy} \\
(D_\mu)_{xy} &= (\partial^x_\mu - ieA_\mu(x))\delta^d(x-y) \\
\Delta^*_{xy} &= \int (dz)\, (D^*_\mu)_{xz}(D^*_\mu)_{zy} \\
(D^*_\mu)_{xy} &= (\partial^x_\mu + ieA_\mu(x))\delta^d(x-y)
\end{aligned} \tag{7.3}$$

and we will choose $\alpha Z = \partial \cdot A$ as above.

Further, to check the finiteness and gauge invariance of the system, we compute, as in Section 6, the $d=4$ one-loop photon mass using Langevin techniques. We first need the integral form of the Langevin system

$$A_\mu(x, t) = \int (dy) \int_{-\infty}^{t} dt'\, G_{\mu\nu}(x-y, t-t') \times \Big[-ie\phi^*(y, t')(\vec{\partial}_\nu - \overleftarrow{\partial}_\nu)\phi(y, t') - 2e^2\phi^*(y, t')\phi(y, t')A_\nu(y, t') + \int (dz)\, K_{xy}(\Box)\eta_\nu(z, t') \Big] \tag{7.4}$$

$$\phi(x, t) = \int (dy) \int_{-\infty}^{t} dt'\, G(x-y, t-t') \times \Big[-ieA_\mu(y, t')\,\partial_\mu\phi(y, t') - ie\partial_\mu(A_\mu(y,t')\phi(y, t')) + ie\frac{1}{\alpha}\,\phi(y, t')\,\partial_\mu A_\mu(y, t') - e^2\phi(y, t')A_\mu(y, t')A_\mu(y, t') + \int (dz)\, K_{yz}(\Delta)|_{t'}\eta(z, t') \Big] \tag{7.5}$$

with a similar equation for ϕ^*. Here

$$G_{\mu\nu}(x-y, t-t') = \Theta(t-t') \int (dp)\, e^{-ip(x-y)} \times [T_{\mu\nu}(p)\, e^{-p^2(t-t')} + L_{\mu\nu}(p)\, e^{-p^2(t-t')/\alpha} \tag{7.6a}$$

$$G(x-y, t-t') = \Theta(t-t') \int (dp)\, e^{-ip(x-y)}\, e^{-(p^2+m^2)(t-t')} \tag{7.6b}$$

are the photon and scalar Langevin Green functions, respectively.

The first step in a weak coupling expansion of (7.4) and (7.5) is the expansion of the charged scalar form factor to the desired order, which is given by formula (5.12) in Section 5. There $g \to e$,

$$\begin{aligned} (\Gamma_1)^{ab}_{xy} &\Rightarrow (\Gamma_1)_{xy} = -i(\partial^x_\mu A_\mu(x) + A_\mu(x)\partial^x_\mu)\delta^d(x-y)l^2 \\ (\Gamma_2)^{ab}_{xy} &\Rightarrow (\Gamma_2)_{xy} = -A_\mu(x)A_\mu(x)\delta^d(x-y)l^2 \end{aligned} \tag{7.7}$$

should be changed. As usual, in (7.7) the derivatives act on everything to the right. This expression may be continued to all orders as shown in Figure 16. In the figure, in accordance with the diagrams of Figure 9, each specific line corresponds to the form factors K, $K^{(1)}$, $K^{(2)}$, and $H(p^2l^2)$, and wavy lines correspond to gauge fields, while the three-, four-, and five-point vertices represent Γ_1 and Γ_2. The heavy arrows denote the retarded property of the Langevin Green functions, while the thinner arrows track the direction of the charge flow on scalar lines

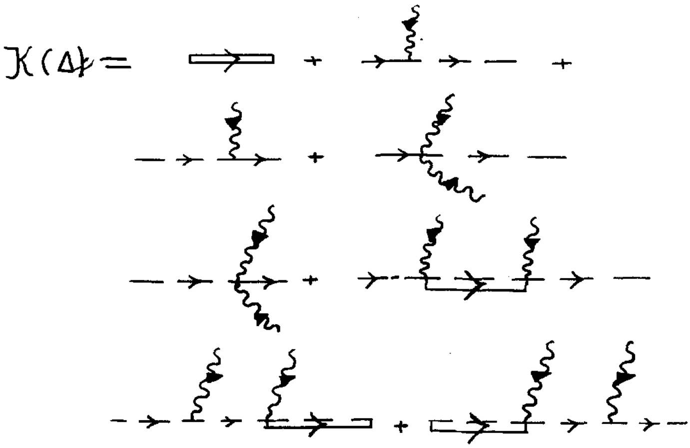

Fig. 16. Expansion of the charged scalar form factor in the nonlocal stochastic scheme.

Having expanded the form factor, an essentially standard (Parisi and Wu, 1981; Bern *et al.*, 1987*b*) iterative procedure allows the expansion of the Langevin solution

$$A_\mu[\eta] = \sum_{m=0}^{\infty} e^m A_\mu^{(m)} \tag{7.8}$$

$$\phi(\eta) = \sum_{m=0}^{\infty} e^m \phi^{(m)}, \qquad \phi^*(\eta) = \sum_{m=0}^{\infty} e^m \phi^{*(m)} \tag{7.9}$$

to any desired order. For the photon mass computation, the relevant results for the form factor $K^{(i)}(-p^2 l^2)$ are

$$A_\mu^{(0)}(x, t) = \int (dy) \int_{-\infty}^{t} dt'\, G_{\mu\nu}(x-y, t-t') \times \int (dz)\, K_{zy}(\Box)\eta_\nu(z, t') \tag{7.10a}$$

$$eA_\mu^{(1)}(x, t) = \int (dy) \int_{-\infty}^{t} dt'\, G_{\mu\nu}(x-y, t-t') \times [-ie\phi^{*(0)}(y, t')(\vec{\partial}_\nu - \overleftarrow{\partial}_\nu)\phi^{(0)}(y, t)] \tag{7.10b}$$

$$\begin{aligned} e^2 A_\mu^{(2)}(x, t) = \int (dy) \int_{-\infty}^{t} dt'\, G_{\mu\nu}(x-y, t-t') &\times [-ie^2\phi^{*(1)}(y, t')(\vec{\partial}_\nu - \overleftarrow{\partial}_\nu)\phi^{(0)}(y, t') \\ &- ie^2\phi^{*(0)}(y, t')(\vec{\partial}_\nu - \overleftarrow{\partial}_\nu)\phi^{(1)}(y, t') \\ &- 2e^2 A_\nu^{(0)}(y, t')\phi^{*(0)}(y, t')\phi^{(0)}(y, t')] \end{aligned} \tag{7.10c}$$

and

$$\phi^{(0)}(x,t)=\int(dy)\int_{-\infty}^{t}dt'\,G(x-y,t-t')\int dz\,K_{yz}(\Box)\eta(z,t'); \qquad (7.11a)$$

$$\begin{aligned}e\phi^{(1)}(x,t)=\int(dy)\int_{-\infty}^{t}dt'\,G(x-y,t-t')\Big\{&-ie[A_\mu^{(0)}(y,t')\,\partial_\mu\phi^{(0)}(y,t')\\&+\partial_\mu(A_\mu^{(0)}(y,t')\phi^{(0)}(y,t'))-\frac{1}{\alpha}\phi^{(0)}(y,t')\,\partial_\mu A_\mu^{(0)}(y,t')]\\&+\frac{e}{2}\int(dz)\,[H(\Box)\Gamma_1K^{(1)}(\Box)+K^{(1)}(\Box)\Gamma_1H(\Box)]_{yz}\eta(z,t')\Big\}\end{aligned} \qquad (7.11b)$$

Such expansions may be represented diagrammatically to all orders as Langevin tree graphs, as shown in Figure 17.

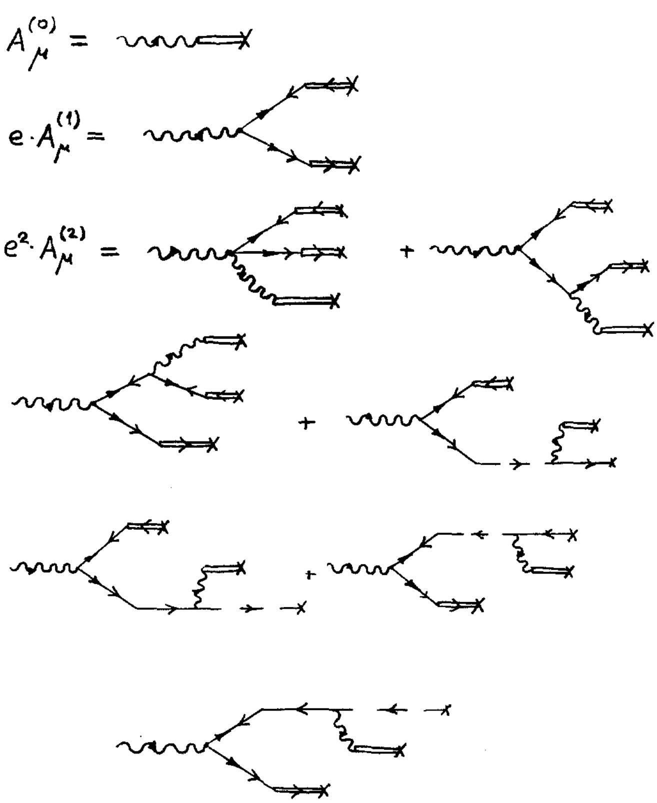

Fig. 17. Langevin tree diagrams for the photon field in the nonlocal stochastic scheme.

According to Bern *et al.* (1987*b*), these Langevin tree diagrams may be constructed to all orders from the simple set of momentum-space Langevin tree rules shown in Figure 18. Finally, the diagrams of the n-point nonlocal Green functions are formed by contracting the trees, as usual, according to equations (7.2a) and (7.2b).

There are three types of diagrams in Figure 19 giving nonvanishing contributions to the zero-momentum vacuum polarization (we do not explicitly exhibit diagrams which are trivially related by symmetry). We now study these diagrams. First, to calculate corresponding contributions,

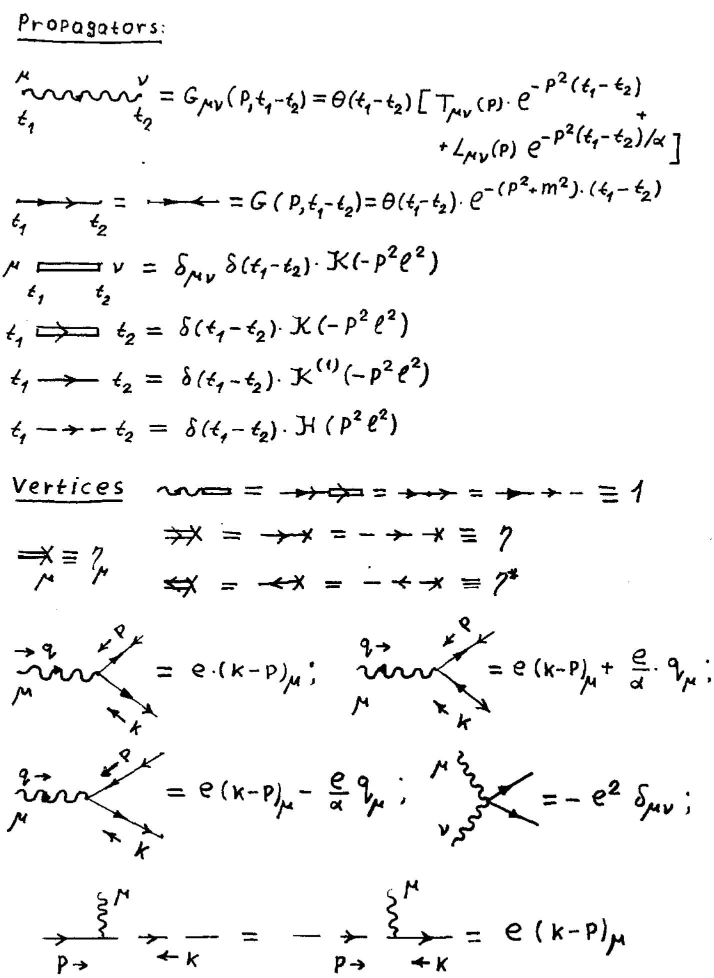

Fig. 18. Langevin tree rules for scalar electrodynamics in the nonlocal stochastic scheme.

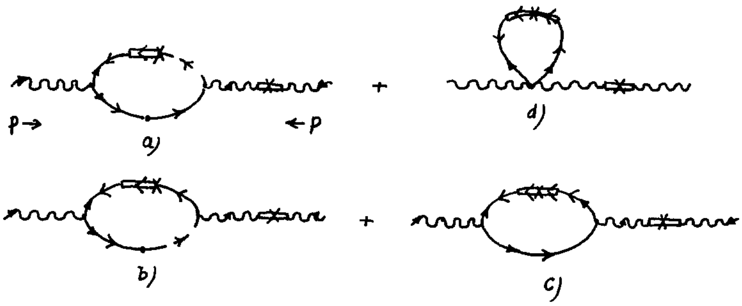

Fig. 19. Nonvanishing contributions to photon mass in the nonlocal stochastic scheme.

an expression for $A_\mu^{(2)}(x, t)$ should be found. In accordance with (7.4), its value in momentum representation acquires the form

$$
\begin{aligned}
A_\mu^{(2)}(p, t) = 2e^2 \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt_1 \int_{-\infty}^{t'} dt_2\, (dp_1)\,(dp_2)\,(dp_3) \\
\times G_{\mu\nu}(p, t-t') G(p_2, t'-t_2) K(-p_1^2 l^2) \\
\times \Bigg\{ \int_{-\infty}^{t_2} dt_3 \int_{-\infty}^{t_2} dt_4\, (dp_4)\, G(p_1, t'-t_1)(p_1+p_2)_\nu \bar{\delta}^d(p+p_1-p_2) \\
\times \bar{\delta}^d(p_2-p_3-p_4) G_{\delta\sigma}(p_3, t_2-t_3) G(p_4, t_2-t_4) K(-p_3^2 l^2) \\
\times K(-p_4^2 l^2) \left[(p_3+2p_4)_\delta - \frac{1}{\alpha} p_{3\delta} \right] \eta^*(p_1, t_1) \eta(p_4, t_4) \\
\times \eta_\sigma(p_3, t_3) + \int_{-\infty}^{t_2} dt_3\, (dp_4)\, G(p_1, t'-t_1)(2p_4+p_3)\rho (p_1+p_2)_\nu \\
\times \bar{\delta}^d(p+p_1-p_2) \bar{\delta}^d(p_2-p_3-p_4) G_{\rho\beta}(p_3, t_2-t_3) K(-p_3^2 l^2) \\
\times \tfrac{1}{2}[H(p_2^2 l^2) K^{(1)}(-p_4^2 l^2) + K^{(1)}(-p_2^2 l^2) H(p_4^2 l^2)] \\
\times \eta^*(p_1, t_1) \eta_\beta(p_3, t_3) \eta(p_4, t_4) \\
- \int_{-\infty}^{t'} dt_3\, G(p_3, t'-t_3) G_{\nu\beta}(p_1, t'-t_1) \\
\times \bar{\delta}^d(p+p_2-p_1-p_3) K(-p_2^2 l^2) K(-p_3^2 l^2) \eta_\beta(p_1, t_1) \\
\times \eta^*(p_2, t_2) \eta(p_3, t_3) \Bigg\} \qquad (7.12)
\end{aligned}
$$

Next, following the methods of Section 6 and using contractions between $A_\mu^{(1)}(x, t)$ and $A_\nu^{(1)}(y, t)[A_\mu^{(2)}(x, t)$ and $A_\nu^{(0)}(y, t)]$, one can obtain the explicit value for the diagrams shown in Figure 19.

Thus, the diagram in Figure 19a gives the following contribution to the photon mass renormalization:

$$\Pi^{(a)}_{\mu\nu}(p) = -8e^2 \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt_1 \int_{-\infty}^{t'} dt_2\,(dp_2)\, G_{\mu\rho}(p, t-t')$$
$$\times G_{\rho\beta}(p, t'-t_1) G(p_2, t'-t_2) G(p_2, t'-t_2)$$
$$\times G_{\nu\delta}(p, t'-t_1)\delta_{\beta\delta} K(-p^2l^2) K(-p_2^2 l^2)$$

After integration over fifth-time variables and truncation near $p=0$, which is accomplished by removal of the two $\Delta_{\mu\rho} = (T^{(p)}_{\mu\rho} + \alpha L^{(p)}_{\mu\rho})/p^2$ factors, we obtain at equilibrium

$$\Pi^{(a)}_{\mu\nu}(0) = -2e^2\delta_{\mu\nu} \int (dq)\, \frac{V(-q^2l^2)}{m^2+q^2} = -2e^2\,\frac{\sigma + m^2e^2 \ln m^2\mu^2}{16\pi^2 l^2}\,\delta_{\mu\nu} \quad (7.13)$$

where we have assumed $V(-p^2l^2)|_{p^2\to 0} = 1$ by the normalization condition and notation $\sigma = \lim_{x\to -1} v(x)/(1+x)$. It is easily seen that contributions corresponding to the diagrams in Figures 19b and 19c are equal to each other, the explicit value of which is given by

$$\Pi^{(b)}_{\mu\nu}(p) = \Pi^{(c)}_{\mu\nu}(p) = -4e^2\, V(-p^2l^2) l^2 \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt_2 \int_{-\infty}^{t'} dt_3\,(dp_1)$$
$$\times G_{\mu\rho}(p, t-t') G(p_1, t'-t_2) G(p+p_1, t'-t_2)$$
$$\times G_{\delta\beta}(p_1, t_2-t_3) G_{\nu\beta}(p, t'-t_3)(2p_1+p)_\rho(2p_1+p)_\delta$$
$$\times K(-p_1^2l^2) K^{(1)}(-(p+p_1)^2l^2) H(p_1^2l^2)$$

Elementary integration over fifth-time variables gives in the limit $p\to 0$

$$\Pi^{(b)}_{\mu\nu}(p) = -e^2\,\Delta_{\mu\rho}(p)\Delta_{\rho\nu}(p) l^2 V(-p^2l^2)$$
$$\times \int (dq)\, \frac{q^2 K(-q^2l^2) K^{(1)}(-q^2l^2) H(q^2l^2)}{m^2+q^2} \quad (7.14)$$

or

$$\Pi^{(b)}_{\mu\nu}(0) = \frac{e^2}{16\pi^2}\,\frac{\sigma}{2l^2}\,\delta_{\mu\nu}$$

Finally, the contribution corresponding to the diagram of figure 19d is

$$\Pi^{(d)}_{\mu\nu}(p) = -8e^2 V(-p^2l^2) \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt_2 \int_{-\infty}^{t_2} dt_3 \int_{-\infty}^{t_2} dt_4\,(dq)\, G_{\mu\rho}(p, t-t')$$
$$\times (2q+p)_\rho G(p_1, t'-t_4) G(p+q, t'-t_2)\left[(2q+p)_\delta - \frac{1}{\alpha} p_\delta\right]$$
$$\times G_{\delta\sigma}(p, t_2-t_3) G_{\sigma\nu}(p, t'-t_3) G(q, t_2-t_4) V(-q^2l^2)$$

Here, integrations over fifth-time variables and d^4q should be carried out, and the result reads in the limit $p \to 0$

$$\Pi^{(d)}_{\mu\nu}(0) = \frac{e^2}{16\pi^2 l^2}(\sigma + 2m^2 l^2 \ln l^2\mu^2)\delta_{\mu\nu} \tag{7.15}$$

The reader may easily verify that the sum of all contributions is zero, so the photon remains massless to this order for the nonlocal stochastic quantization theory, as it should.

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